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# Quasi Cancellation R-Modules

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**ABSTRACT**

Let R be a commutative ring with identity, and let M be a unitary left R-module. In this work, we introduce quasi cancellation R-module (weakly quasi cancellation R-module) concept as a generalization of cancellation R-module (weakly cancellation R-module) concept we generalize some properties of cancellation (weakly cancellation) R-modules to quasi cancellation (weakly quasi cancellation) R-modules. Furthermore, we give some conditions under which R-module M is quasi cancellation R-module.

## Introduction

Let R be a commutative ring with identity, and let M be a unitary left R-module. Gilmer [7] has been defined the concept of cancellation ideal to be the ideal of R which satisfies the following: whenever AI=BI with A and B are ideals of R, implies = B . Mijbass in [12] has been generalized this concept to modules. He has been defined the cancellation module as follows: An R- modudule M is called a cancellation module, whenever AM=BM with A and B are ideals of R, implies A=B.

This thesis include four sections .In section one, we introduce quasi cancellation R-modudule concept. An R-module M is said to be quasi cancellation, whenever EM=ABM, with E, A and B are ideals of R, implies either E=A or E=B, or both. The class of cancellation R-modules is different from the class of quasi cancellation R-modules. Moreover, we give example which indicate that two classes are different indicate that two, however we put some conditions under which the two classes are eguivalent. We prove that if R is Boolean ring and M is quasi cancellation R-module then M is cancellation (Remark1-3). If M is cancellation R-module and EM=ABM, with E, A and B are ideals of R, then M is quasi cancellation if A or B is an identity ideal (Remark 1-4). And if A ≤ B or B ≤ A (prop. 1-5). E is prime ideal (Prop.1-6), so E is maximal ideal (Prop. 1-8). Next, we prove that an R-module M is quasi cancellation if it is generated by non torsion element and whenever EM=ABM with A and B are ideals of R. E is prime ideal of R (Th. 1-9).

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In section two, we define quasi cancellation submodules, so we study the relation between quasi cancellation submodules and quasi cancellation R-modules. We give conditions under which a quasi cancellation submodule N becomes quasi cancellation R-module M. if M is a multiplication (Prop.2-2) and if N is a pure submodule, see (Prop.2-3) . Moreover, we give example which indicate the converse is not true in general, thus we put another conditions as we see in (Prop. 2-6) and (corollary 2-8). In section three, we introduce a generalization for cancellation ring concept namely quasi canellation ring, and we study the relation between them (Remark 3-2). Also, we define quasi canellation ideals. We give some conditions under which quasi cancellation submodules IM of multiplication R-module becomes quasi cancellation ideal I of R (Prop. 3-5(1)). But the converse is not true in general , thus we put a condition that M is a cancellation R-module (prop. 3-5(2)). We end this section by characterizations of quasi cancellation ideals, see (Th. 3-7). Finally, we generalize weakly cancellation R-modules to weakly quasi cancellation R-modules. An R-module M is said to be weakly cancellation, whenever AM=BM with A and B are ideals of R, implies A+ann(M)=B+ann(M) [14] .

In this section , we define weakly quasi cancellation R-module M , whenever EM=ABM with E , A and B are ideals of R, implies either E+ann(M)=A+ann(M) or E+ann(M)=B+ann(M). Also, we discuss the validity of the results that we obtain in this section , we show that if R is arithmetical ring , then the class of cyclic R-modules is contained in the class of weakly quasi cancellation R-module (Prop. 4-4) .Next , we give some characterizations of weakly quasi cancellation R-modules (Th. 4-5).

**S1 :- The relationships between cancellation modues and quasi cancellation modules .**

In this section we introduce the definition of quasi cancellation R-modules, with some examples about this concept. Moreover we establish some relationships between cancellation R-modules and quasi cancellation R-modules.

**1-1 Definition**

An R-module M is called quasi cancellation whenever EM = ABM with E, A and B are ideals if a ring R , implies either E = A or E = B .

**1-2 Examples and Remarks**

1/ Z4 is a quasi cancellation Z8-module , since <2> Z4 = <2> <4> Z4
and <4> Z4 = <2> <4> Z4 .

2/ Z6 is not a quasi cancelation Z12-module , since <6> Z6 = <3> <4> Z6 . But <6>≠<3> and <6>≠<4> .

3/ Z12 is not a quasi cancellation Z24-module , since <2> Z12 = <4> <6> Z12 . But <2>≠<4> and <2>≠<6> .

4/ Clearly, the class of cancellation modules is deferent from the class of quasi cancellation modules. However, we shall give a sufficient conditions under which the two classes are equivalent.
The following example show that is not necessary that each cancellation module is quasi cancellation module.
5/ Z3 is a quasi cancellation Z6–module. But it is not cancellation, because <2> Z3 = <3> Z3 , but <2>≠<3> .

The following remark gives a necessary condition to get cancellation modules from quasi cancellation modules .

Before we give the following proposition , we need the following definitions .

Recall the a ring R is called a Booldean ring in case each of its elements is idempotent , [6] .

Recall that an ideal I of a ring R is called idempotent if I2 = I , [11] .

**1-3 Remark**

Let R be a Boolean ring and let M be an R-module If M is a quasi cancellation R-module, then M is a cancellation R-module .

**Proof :**

Let M be a quasi cancellation R-module , and let AM = BM with A and B are ideals of R . Since R is a Boolean ring, then AM = BM = $B^2$ M = BBM. And since M is a quasi cancellation R-module , then A=B . Thus M is a cancellation R-module.

The following results gives sufficient conditions under which the cancellation modules is quasi can collation modules.

**1-4 Remark**

Let M be an R-module and let $EM = ABM$ with E , A and B are ideals of R , such that either A or B is the identity ideal . If M is a cancellation R-module , then M is a quasi cancellation R-module .

**Proof :**
Clear

Recall that an ideal I of a ring R is a prime if for each a , b ∈ R , a b ∈ I, then either a ∈ I or b ∈ I , [3] .

The following propositions give another condition to get quasi cancellation modules from cancellation modules .

**1-5 proposition**

Let M be an R-module , and let EM = ABM with E , A and B are ideals of R , such that either A ⊆ B or B ⊆ A . If M is a cancellation R-module , then M is a quasi cancellation R-module .

**Proof :**

Let EM = ABM with E , A and B are ideals of R . If A ⊆ B , then AB = B . Hence EM = BM and since M is a cancellation R-module , then E = B. Similarly , if B ⊆ A , we can get that E = A . Therefore M is a quasi cancellation R-module .

**1-6 Proposition**

Let M be R-module , and let EM = ABM with E is a prime ideal of R . A , B are ideals of R . If M is a cancellation R-module , then M is a quasi cancellation R-module .

Proof : Let EM=ABM with E is a prime ideal of R . A and B are ideals of R . Since M is a cancellation R-module , then E = AB , and since E is a prime ideal of R , them either E = A or E = B . Thus M is a quasi cancellation R-module .

The converse is not true in general , as we see in the following example .

**1-7 Example**

Z4 as Z8-module is not cancellation , but it is a quasi cancellation . Since <2> Z4 = <2> <4> Z4 , and <2> Z4 = <4> Z4 . But <2>≠<4> .

Recall that a proper ideal I of a ring R is said to be a maximal ideal of R , if there exists an ideal J of R such that I⊆J⊆ R , then J = R , [1] .

**1-8 Proposition**

Let M be an R-module , and let EM = ABM with E is a maximal ideal of R . A and B are ideals of R . If M is a cancellation R-module , then M is a quasi cancellation R-module .

Proof : Since R is a commutative ring with identity , and since E is a maximal ideal of R . Hence R/ H is a field , so in particular it is an integral domain , [9] .

Thus E is a prime ideal of R , [11] . So by prop.(1-5) we get M is a quasi cancellation R-module .

The converse is not true in general , as we see in the review example .

Recall that an ideal I of a ring R is said to be irreducible if for each ideals A and B of R with I = A ∩ B , implies I = A or I = B, [11] .

The following theorem gives a sufficient condition under which R-module is quasi cancellation . First , we needed the following definition , which appear in , [10] .

Recall that an element $m \in M$ ,such that M is an R-module is a torsion if there exists $0 \neq r \in R$, such that $rm = 0$.

**1-9 Theorem**

Let M be an R-module generated by a non torsion element , and let EM = ABM with E is a prime ideal of R . A and B are ideals of R . Then M is a quasi cancellation R-module .

Proof : Let $M = (m)$, such that m is a non torsion element of M . And let EM = ABM with A and B are ideals of R . E is a prime ideal of R , then $E(m) = AB(m)$, hence $xm \in AB(m)$ for each $x \in E$ there exists $y \in AB$, such that $xm = ym$ , hence $(x - y)m = o$ . Since m is a non torsion element , thus $x - y = o$ so $x = y \in AB$ . Hence $\subseteq AB$ . So we can get the converse by the same method . Therefore E = AB , and since E is a prime ideal of R , thus either E=A or E=B . Therefore M is a quasi cancellation R-module .

In the following theorem , we give some characterizations of quasi cancellation modules .

**1-10 theorem**

If M is an R-module generated by a non torsion element , then the following statements are equivalent .

1/ M is a cancellation R-module if EM = ABM , with E is a prime ideal of
R . A and B are ideals of R .

2/ M is a quasi cancellation R-module .

3/ If $(x)M = ABM$ , where A and B are ideals of R , and $x \in R$ . then
either $x \in A$ or $x \in B$ .

4/ $E = (EM{:}M)$ , for each ideal E of R .

**Proof :**

(1) ⇒ (2) By prop.(1-7)

(2) ⇒ (3) Let $(x)M = ABM$ , where A and B are ideals of R , and $x \in R$ . Since M is a quasi cancellation R-module , then either $(x) = A$ or $(x) = B$ . If $(x) = A$ implies $x \in A$ , and if $(x) = B$ implies $\in B$ .

(3) ⇒ (4) Let $x \in E$ , implies $\subseteq EM$ , hence $x \in (EM = M)$ , thus $E \subseteq (EM{:}M)$ . Now let $y \in (EM{:}M)$, implies $yM \subseteq EM$ . Since M is generated by a non torsion element , thus let $M = (m)$, such that M is a non torsion element of M . Then $y(m) \subseteq E(M)$ , there exists $a \in E$, such that $ym = am$ , so $(y - a)m = o$, thus $y = a \in E$ . Therefore $(EM{:}M) \subseteq E$, hence $E = (EM{:}M)$.

(4)⇒ (1)Let EM=AM , where E and A are ideals of R . then $E \subseteq (AM{:}M)$ , and by(4) A=(AM:M) . Thus $E \subseteq A$ . So from $= EM$ , implies $A \subseteq (EM{:}M) = E$ , hence $E =$ A. Therefore M is a cancellation R-module , which completes the proof .

**S2 : Quasi cancellation submodules**

In this section we introduce the definition of quasi cancellation submodules , with some results show the relationship between quasi cancellation submodules and quasi cancellation modules .

**2-1 Definition**

A submodmle N of an R-module M is said to be a quasi cancellation R-submodule , if EN = ABN, where E , A and B are ideals of R . Then either = A or E = B .

Recall that an R-module M is called a multiplication if for each submodule N of M , there exists an ideal I of R such that N = IM , [13] .

**2-2 Proposition**

Let M be a multiplication R-module , and let N be a submodule of M . If N is a quasi cancellation R-submodule then M is a quasi cancellation R-module .

**Proof :**

Let N be a quasi cancellation R-submodule of M , since M is multiplication R-module , thus there exists an ideal I of R , such that N = IM . Let $EM = ABM$, where E , A and B are ideals of R . Now EIM = ABIM , thus $EN = ABN$. Since N is a quasi cancellation R-submodule , hence either E = A or E = B . Therefore M is a quasi cancellation R-submodnle .

The following propositions gives some conditions to get the converse of the review proposition .

Recall that a submodule N of an R-modnle M is said to be pure if $N \cap IM = IN$, for each ideal I of R, [5] .

**2-3 proposition**

Let M be an R-module , and N is a pure submodule of M . If N is quasi cancellation submodule , then M is a quasi cancellation module .

**Proof :**

Let $EM = ABM$, where E , A and B are ideas of R . Since N is a pure submodule of M , than $N \cap EM = EN$ and $N \cap ABM = ABN$, hence $EN = ABM$, and since N is a quasi cancellation submodule , implies either E = A or E = B . Therefore M is a quasi cancellation module .

The converse is not true in general , as we see in the following example

**2-4 Example**

Z9 as Z18–module is a quasi cancellation module . Z3 is a submodule of Z9 , and Z3 is not a quasi cancellation submodule of Z9 . Because (3)Z3 =(2)(6)Z3 , but $(3) \neq (2)$ and $(3) \neq (6)$.

Recall that a proper submodule N of an R-module M is said to be a maximal submodule if there exists a submodule L of M such that $N \subset L \subseteq M$, then L = M , [1] .

Before starting the next result ,we will need the following lemma ,which is the key of it .

**2-5 lemma**

Let N be a maximal submodule of an R-module M . If $L \cap N = K \cap N$ for each submodules L and K of M , then L = K .

**Proof :**

Let $x \in L$ , then $x \in N$ (because N is a maximal submodule of M ) . Thus $x \in L \cap N$ , so $x \in K$ , since $L \cap N = K \cap N$, implies $x \in K \cap N$, so $x \in K$, therefore $L \subseteq K$ . Simillarly , we can proof that $K \subseteq L$ , which is implies $L = K$.

**2-6 proposition**

Let M be a multiplication R-.module , and let N be a maximal submodule of M . If M is a quasi cancellation module and N is a pure submodule , then N is a quasi cancellation submodule . **proof :**
Let EN = ABN, where E , A and B are ideals of R . Since N is a pure submodule , then $N \cap EM = EN$ and $N \cap ABM = ABN$ , hence $N \cap EM = N \cap ABM$ . Since M is a multiplication R-module and N is a maximal submodule , thus by lemma(2-5) we can get EM = ABM , and since M is a quasi cancellation module , implies either E = A or E = B . Therefore N is a quasi cancellation submodule .

The following corollaries follow directly from the previews proposition

**2-7 Corollary**

Let N be a maximal submodule of a multiplication R-module M . If N is a pure submodule . Then N is a quasi cancellation module if and only if M is a quasi cancellation R-module .

Recall that an R-module M is said to be a cyclic if and only if there exists $x \in M$ sach that M = Rx , [1] .

**2-8 Corollary**

Let M be a cyclic R-module and let N be a submodule of M such that $N \cap rM = rN$, for each $r \in R$ . If N is quasi cancellation submodule , then , M is quasi cancellation module .

Proof :

Since M is a cyclic R-module and since $N \cap rM = rN$ , for each $r \in R$ . Them N is a pure submodule of M [4] , and according to the proposition (2-3) , we obtain the result .

**2-9 Corollary**

Let M be acyclic R-module , and let N be a maximal submodule of M such that $N \cap rM = rN$, for each $r \in R$ . Then N is quasi cancellation submodule if and only if M is quasi cancellation module .

An R-module M is called a regular if each submodule of M is pure ,[5] .

**2-10 Corollary**

Let M be a regular R-module and let N be a submodmle of M . If N is quasi cancellation submodmle , then M is quasi cancellation module .

**2-11 Corollary**

Let N be a maximal submodule of a multiplication R-module M . If M is a regular R-module , the N is a quasi cancellation submodule if and only if M is quasi cancellation module .

**S3 : Quasi cancellation rings**

[2] has been defined the concept of cancellation rings to be the ring R which satisfies the following : whenever $sr = t$ , where r , s , $t \in R$ and $r \neq o$ , then $s = t$ .

In this section we introduce a generalization for cancellation ring concept namely a quasi cancellation ring .

**3-1 Definition**

A ring R is said to be a quasi cancellation ring if and only if whenever r , s , a , $b \in R$ , $r \neq o$ and $sr = abr$ , then either $s = a$ or $s = b$ .

Equivalent ; if whenever $E = AB$ , where $E$ , $A$ and B are ideals of R , then either $E = A$ or $E = B$ .

**3-2 Remarks**

1/ If M is a quasi cancellation R-module , then R is a quasi cancellation ring .

2/ If M is a cancellation R-module , and R is a quasi cancellation ring .
Then M is quasi cancellation R-module .

**Proof :**

1/ Let $E = AB$ , where E , A and B are ideals of a ring R , then EM = ABM . Since M is a quasi cancellation R-module , then either E = A or E = B . Thus R is a quasi cancellation ring .

2/ Let $EA = ABM$ , where E , A and B are ideals of R , since M is a cancellation R-module , then E = AB , and since R is a quasi cancellation ring , then either $E = A$ or $E = B$. Therefore M is a quasi cancellation R-module .

**3-3 Definition**

An ideal I or a ring R is called quasi cancellation ideal if $EI = ABI$, where E , A and B are ideals of . Then either $E = A$ or $E = B$.

The following corollary follows directly from the previous remarks .

**3-4 Corollary**

If M is a cancellation R-module , then M is quasi cancellation R-module if and only if R is quasi cancellation ring .

**3-5 Proposition**

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Let M be a multiplication R-module and let I be an ideal of a ring R .

Then

1/ If IM is quasi cancellation submodule , then I is quasi cancellation ideal of R .

2/ If M is cancellation R-module and I is quasi con collation ideal , then IM is quasi cancellation submodule .

**Proof :**

1/ Let $EI = ABI$ , where E , A and B are ideals of a ring R , then EIM=ABIM . Since IM is a quasi cancellation submodmle , implies either E=A or E=B . Therefore I is a quasi cancellation ideal .

2/ Let $EIM = ABIM$, where E , A and B are ideals of R . Since M is a cancellation R-module , then $EI = ABI$ , and since I is a quasi cancellation ideal , implies either $E = A \, or \, E = B$ . Therefore IM is quasi cancellation submodule .

The following corollary follow directly from the previous proposition .

**3-6 Corollary**

Let M be a multiplication R-module , and let I be an ideal of R , If M is a cancellation R-module . Then IM is a quasi cancellation submodule of M if and only if I is quasi cancellation ideal of R .

Next , the following theorem gives the relationship between quasi cancellation submodule and quasi cancellation ideal .

**3-7 Theorem**

Let M be a multiplication R-module and let N be a proper submodule of M , If M is a cancellation R-module .Then the following statements are equivalent .

1/ N is a quasi cancellation submodule of M .

2/ $(N{:}M)$ is a quasi cancellation ideal of R .

3/ N = IM , where I is a quasi cancellation ideal of R .

**Proof :**

(1) ⇒ (2) Let , N be a quasi cancellation submodule of M and let $E(N{:}M) = AB(N{:}M)$ , where E , A and B are ideals of R . Then $E(N{:}M)M = AB(N{:}M)$M , implies that $EN = ABN$. Since N is a quasi cancellation submodule , thus either E = A or E = B . Hence (N:M) is a quasi cancellation ideal of R .

(2) ⇒ (3) We can get that I is a quasi cancellation ideal of R , when $N = IM$ only when put $I = (N{:}M)$ .

(3) ⇒ (1) From prop. (3-5) .

**S4 : weakly quasi cancellation R-modules**

In this section we introduce a generalization for weakly cancellation R-modules concept namely weakly quasi cancellation R-modules .

**4-1 Definition**

An R-module M is called weakly quasi cancellation if $EM = ABM$. where E , A and B are ideals of R . Then either $E + ann(M) = A + ann(M)$ or $E + ann(M) = B + ann(M)$ or both .

So a ring R is called weakly quasi cancellation if $E = AB$ , where E , A and B are ideals of R , then either $E + ann(M) = A + ann(M)$or $E + ann(M) = B + ann(M)$ or both .

**4-2 Remarks**

1/ If M is weakly quasi cancellation R-module , then R is quasi cancellation ring .

2/ Cleary , each quasi cancellation R-module is weakly quasi cancellation R-modules .

Recall that an R-module M is said to be a faithful if ann$(M) = 0$ , where ann$(M) = \{r \in R : rm = o \ \forall m \in M\}$ , [11] .

3/ If M be a faithful R-module . Then M is weakly quasi cancellation R-module if and only if M is a quasi cancellation R-module .

Recall that a ring R is called arithmetical of for each ideals A , B and C of R , we have $(A + B) \cap C = (A \cap C) + (B \cap C)$. This property is equivalent to the condition that $(A \cap B) + C = (A + C) \cap (B + C)$ , [8] .

The following proposition show the relation between weakly cancellation modules and weakly quasi cancellation module .

**4-3 Proposition**

Let R be arithmetical ring and let M be an R-module . If M is weakly cancellation R-modules , then M is weakly quasi cancellation R-modules .

**Proof :**

Let $EM = ABM$, where E , A and B are ideals of R . Since M is weakly cancellation R-modules , then $E + ann(M) = AB + ann(M)$ . Hence $E \subseteq AB + ann(M) \subseteq (A \cap B) + ann(M)$. And since R is arithmetical ring , then $E \subseteq (A + ann(M) \cap (B + ann(M)$ . Thus $E \subseteq A + ann(M)$ and $E \subseteq B + ann(M)$ , so $E + ann(M) \subseteq A + ann(M)$ and $E + ann(M) \subseteq B + ann(M)$ . Since $E + ann(M) = AB + ann(M)$ , then $AB \subseteq E + ann(M)$ . And since $A \subseteq AB$ , so $A \subseteq E + ann(M)$. And $A + ann(M) \subseteq E + ann(M)$ , implies $E + ann(M) = A + ann(M)$ . Therefore M is weakly quasi cancellation R-module .

**4-4 Proposition**

Let R be arithmetical ring , and let M be acyclic R-module then M is weakly quasi cancellation R-modules .

**Proof :**

Let $EM = ABM$ , where E , A and B are ideals of R and let $M = (m)$ such that $o \neq m \in M$. then $xm \in AB(m)$ for each $x \in E$ . So xm = abm for some $a \in A$ and $b \in B$. Hence $(x - ab)m = o$ , implies $x - ab = o$ , so $x - ab \in ann(M)$ . But $x = ab + x = ab$ , thus $x \in AB + ann(M)$ , implies $E \subseteq AB + ann(M) \subseteq (A \cap B) + ann(M)$ . Since R is arithmetical ring , then $(A \cap B) + ann(M) = (A + ann(m) \cap (B + ann(M)$ , implies $E \subseteq (A + ann(M) \cap (B + ann(M)$ . Thus we get $E \subseteq A + ann(M)$ and $E \subseteq B + ann(M)$, so $E + ann(M) \subseteq A + ann(M)$ and $E + ann(M) \subseteq B + ann(M)$ . Now from $EM = ABM$, we get $AB \subseteq E + ann(M)$ . So $A \subseteq AB \subseteq E + ann(M)$ , hence $A + ann(M) \subseteq E + ann(M)$ , implies $E + ann(M) = A + ann(M)$. Therefore M is weakly quasi cancellation R-module .

**4-5 Theorem**

Let R be arithmetical ring and let M be an R-module . Then the following statements are equivalent

1/ M is weakly quasi cancellation R-module .

2/ If $EM \subseteq ABM$, where $E, A$ and B are ideals of R , then $E \subseteq B + ann(M))$ and $E \subseteq B + ann(M)$ .

3/ If $(x)M \subseteq ABM$, where A and B are ideals of R , and $x \in R$ , then $x \in A + ann(M)$ and $x \in B + ann(M)$ .

4/ $(EM{:}M) = E + ann(M)$, for each ideal E of R .

**Proof :**

(1) ⇒ (2) Let $EM \subseteq ABM$ , where E , A and B are ideals of R , $ABM = EM + ABM = (E + AB)$ M , since M is weakly quasi cancellation R-module . Then either $A + ann(M) = E + AB) + ann(M)$ , or $B + ann(M) = (E + AB) + ann(M)$ . Since $A \subseteq AB$ and $B \subseteq AB$ , then $E \subseteq AB + ann(M) \subseteq (A \cap B) + ann(M) = \left(A + ann(M)\right) \cap (B + ann(M)$(*because* Ris arithmetical ring) , thus we get $E \subseteq A + ann(M)$ and $E \subseteq B + ann(M)$ .

(2) ⇒ (3) Let $(x)M \subseteq ABM$, where A and B are ideals of R and $x \in R$ . By (2) we get $(x) \subseteq A + ann(M)$ and $(x) \subseteq B + ann(M)$ . Hence $x \in A + ann(M)$ and $x \in B + ann(M)$ .

(3) ⇒ (4) let $x \in (EM{:}M)$ , where E is an ideal of R , then $xM \subseteq EM$ , so $x \in E + ann(M)$ , hence $(EM{:}M) \subseteq E + ann(M)$ . Now let $y \in E + ann(M)$ , then $yM \subseteq EM$ , so $y \in (EM{:}M)$ , hence $E + ann(M) \subseteq (EM{:}M)$ . Therefore $(EM{:}M) = E + ann(M)$ .

(4) ⇒ (1) let $EM = ABM$ , where E , A and B are ideal of R . $EM \subseteq ABM$, by(2) we get $E \subseteq A + ann(M)$ and $E \subseteq B + ann(M)$ , so $E + ann(M) \subseteq A + ann(M)$ and $E + ann(M) \subseteq B + ann(M)$ . From $EM = ABM$ , we get $ABM \subseteq EM$ . Since $AM \subseteq ABM$ and $\subseteq ABM$ , thus $A \subseteq E + ann(M)$ and $B \subseteq E + ann(M)$ , so $A + ann(M) \subseteq E + ann(M)$ and $B + ann(M) \subseteq E + ann(M)$ . Therefore $E + ann(M) = A + ann(M) and\ E + ann(M) = B + ann(M)$ , hence M is weakly quasi cancellation R-module .

Recall that an R-module M is called prime if and only if $ann(M) = ann(N)$ for every non-zero submodule N of M , [3] .

**4-6 Proposition**

Let N be a pure submodule of an R-module M.

1/ If N is weakly quasi cancellation submodule , then M is weakly quasi cancellation R-module .

2/ Let M be a Prime R-module and let N be a maximal submodule of M . If M is weakly quasi cancellation R-module , then N is weakly quasi cancellation submodule of M .

**Proof :**

1/ Let $EM = ABM$ , where E , A and B are ideals of R . Since N is pure submodule of M , then $N \cap EM = EN$ and $N \cap AB = ABN$. Hence $EN = ABN$, since N is weakly quasi cancellation submodule , then either $E + ann(N) = A + ann(N) or\ E + ann(N) = B + ann(N)$ , so either$E + ann(M) = A + ann(M) or\ E + ann(M) = B + ann(M)$ . Therefore M is weakly cancellation R-module .

2/ Let $EN = ABN$ , where E , A and B are ideals of R . Since N is pure submodule of M , then $N \cap EM = EN$ and $N \cap ABM = ABN$ . Hence $N \cap E\ M = N \cap ABM$, and since N is maximal submodule , then by lemma (2-5) we get $EM = ABM$, and since M is weakly quasi cancellation R-module . Then either E+$ann(M) = A + ann(M)\ or\ E + ann(M) = B + ann(M)$ . Since M is prime R-module , thus either $E + ann(N) = A + ann(N)\ or\ E + ann(N) = B + ann(N)$ . Therefore N is weakly quasi cancellation R-module .

**4-7 Proposition**

Let R be arithmetical ring and let M be an R-module . Then M is weakly quasi cancellation R-module if and only if $(A + ann(M){:}E) = (AM{:}EM)$ and $(B + ann(M){:}E) = (BM{:}EM)$ , where E , A and B are ideals of R .

**Proof :**

Let M be a weakly quasi cancellation R-module and let $x \in (A + ann(M){:}E)$ , then $xE \subseteq A + ann(M)$ , so $xEM \subseteq AM$ . Hence $x \in (AM{:}EM)$ , thus $(A + ann(M){:}E) = (AM{:}EM)$ . Now let $y \in (AM{:}EM)$ , then $yEM \subseteq AM$ . By th. (4.5) we get $yE \subseteq A + ann(M)$, hence $y \in (A + ann(M){:}E)$ . Then $(AM{:}EM) \subseteq (A + ann(M){:}E)$, so $(AM{:}EM) = (A + ann(M){:}E)$ . Similarly , we can prove that $(BM{:}EM) = (B + ann(M){:}E)$ .

Conversely ; Let $EM = ABM$, where E , A and B are ideals of R , so $EM \subseteq ABM$ thus $(ABM{:}EM) = R$, hence $(AB + ann(m){:}\,E) = R,$ and $E \subseteq AB + ann(M) \subseteq (A \cap B) + ann(M) \subseteq (A + ann(M)) + (B + ann(M))$(because R is arithmetical ring) . Thus $E \subseteq A + ann(M)$ $and\ E \subseteq B + ann(M)$ . By th. (2-5) we get M is weakly quasi cancellation R-module .

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# موديولات الحذف الكاذب


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**الخلاصة :**

لتكن $R$ حلقة ابدالية ذات عنصر محايد، وليكن $M$ موديول احادي ايسر معرف على الحلقة $R$ . أن الهدف من هذا البحث هو تعميم مفهوم موديولات الحذف إلى موديولات حذف كاذبة كما تم تعميم مفهوم موديولات الحذف الضعيفة إلى موديولات الحذف الكاذبة الضعيفة. كما درسنا العلاقة بينهما وكيف يمكن الحصول على موديولات الحذف الكاذبة من موديولات الحذف وبالعكس وكذلك قام البحث بتعميم بعض خصائص موديولات الحذف على موديولات الحذف الكاذبة وذلك بوضع بعض الشروط الضرورية .