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PERIODIC SOLUTION FOR A CLASS OF DOUBLY DEGENERATE PARABOLIC EQUATION WITH NEUMANN PROBLEM

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ABSTRACT

In this article, we study the periodic solution for a class of doubly degenerate parabolic equation with nonlocal terms and Neumann boundary conditions. By using the theory of Leray-Schauder degree, we obtain the existence of nontrivial nonnegative time periodic solution.

1. INTRODUCTION

The goal of the present text is to consider the boundary conditions in equations (1.1) to (1.3) for periodic doubly degenerate parabolic equation with Neumann boundary.

$$\frac{\partial u}{\partial t} - div(|\nabla u^m|^{p-2}\,\nabla u^m) = (m - \phi[u])\,u, \to \quad (x,t) \in Q_T \qquad 1.1$$

$$\frac{\partial u}{\partial t} = 0, \qquad (x,t) \in \partial\Omega \times (0,T) \qquad 1.2$$

$$u(x,0) = u(x,T), \qquad x \in \Omega \qquad 1.3$$

Where m ≥ 1, p ≥ 2, the habitat Ω is a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$, $\frac{\partial}{\partial n}$ denotes the outward normal derivative on $\partial\Omega$. The zero-flux boundary condition in equation (1.2) means that no individuals cross the boundary of the habitat, $Q_T = \Omega \times (0, T)$. This problems is motivated by models which have been proposed for some problems in mathematical biology. The unknown function *u(x,t)* depends on both location of *x* and time *t*, and the diffusion term $div(|\nabla u|^{p-2}\,\nabla u)$, $(\Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i})$ models the tendency to avoid high density in the habitat. As population growing is controlled by birth, death, emigrant, and immigration, assumption of m, $\Phi[u]$ hould be made to describe the ways in which a given population grows and shrinks over time.

Recently, periodic problems with nonlocal terms have been investigated intensively by number of researchers [1–5]. A typical model was submitted by Allegretto and Nistri in which they proposed the following equations:

$$\frac{\partial u}{\partial t} - \Delta u = f(x,t,a,\Phi[u],u)$$

with Dirichlet boundary conditions. Also, according to the actual needs, many authors diverts attention to nonlinear diffusion equations with nonlocal terms such as the porous equation [6, 7] with typical form:

$$\frac{\partial u}{\partial t} - \Delta u^m + (a - \Phi[u])u \qquad 1.4$$

And a class double degenerate parabolic equation [8] with the typical from shown in equation (1.5).

$$\frac{\partial u}{\partial t} - div(|\nabla u^m|^{p-2}\,\nabla u^m) + (a - \Phi[u])\,u \qquad 1.5$$

The equation (1.4) is degenerate if $m > 1$ and singular if 0 < m <1. In addition, equation (1.5) is also degenerate when $u = 0$, or when the gradient of u vanishes. These degenerate equations exhibit a doubly nonlinearity which generalize the porous medium equation $p = 2$ and the parabolic p-Laplace equation m = 1. If p = 1 and m = 1 then equation (1.5) becomes a nondegenerate parabolic equation and heat equation is its special case.

By comparing the doubly degenerate parabolic equation with Dirichlet boundary equation, the Neumann boundary condition causes an additional difficulty in establishing a priori estimate. On the other hand, different form the case of Dirichlet boundary condition, the auxiliary problem in equations (1.1) to (1.3) is considered for using the theory of Leray-Schauder degree. We have proved that the problem in equations (1.1) to (1.3) admits a non-trivial

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nonnegative periodic solution as shown in the following theorem.

The rest of this article is organized as follows: In Section 2, we present some necessary preliminaries including the auxiliary problem. in section 3, we establish the necessary priori estimations of the solution of the auxiliary problem. Then the proof of the main result of this article is shown in the last section.

2. Preliminaries

In this paper, we assume that:

$(B1)\Phi[.]:L_+^2(\Omega)\to R^+$ are the boundary conditions functional satisfying the condition:

$$0\le\Phi[u]\le K\|u\|_{L^2(\Omega)}^2$$

Where K > 0 is constant independent of u, + $\mathbf{R}^+=[0,+\infty), L_+^2(\Omega)=\{u\in L^2(\Omega)\ \mid u\ \ge\ 0, a.\text{ e. } in\ \Omega\}$.

(B2) $m(x,t)\in C_T(\overline{Q}_T)$ and satisfies that $\{x\in\Omega:\frac{1}{T}\int_0^T m(x,t)>0\}\ne\varnothing$, where $C_T(\overline{Q}_T)$ denotes the set of function which are continuous in $(\overline{\Omega}\times \mathbf{R})$ and of T- periodic with respect to t.

From (B2), we can see that there exist $x_0\in\Omega,\delta>0,m_0>0$ such that

$\frac{1}{T}\int_0^T m(x,t)dt\ge m_0,$ for all $x\in B(x_0,\delta).$

Since the equation (1.1) is degenerate at points where $u=0,$ the problem (1.1)-(1.3) has no classical solutions in general, so we focus on the discussion of weak solution in the sense of the following

$$\operatorname*{essinf}_{x\in\Omega}\frac{1}{T}\int_0^T m(x,t)dt>\gamma\lambda_1.$$

Where λ_1 is the first eigenvalue of the Laplacian equation on T with zero boundary and $\phi_1(x)$ be the corresponding eigenfunction.

Since the regularity follows from a quite standard approach, we focus on the discussion of weak solutions in the following sense.

Definition 1 A function u is said to be a weak solution of the problem (1.1) - (1.3), if $u\in L^\infty(Q_T)\cap C_T(\overline{Q}_T), u^m\in L^p(0,T;W_0^{1,p}(\Omega))\cap C_T(\overline{Q}_T)$ and u satisfies

$$\iint_{QT}(-u\frac{\partial\varphi}{\partial t}+|\nabla u^m|^{p-2}\nabla u^m\nabla\varphi-(m-\phi[u])u\varphi)dxdt=0. \qquad (2.1)$$

For any $\varphi\in C^1(\overline{Q}_T)\ with\ \varphi(x,0)=\varphi(x,t).$

In order to use the theory of Leray-Schauder degree, we introduce a map by considering the following auxiliary problem

$$\frac{\partial u_\varepsilon}{\partial t}-div((|A(u_\varepsilon)\nabla u_\varepsilon|^2+\varepsilon)^{\frac{p-2}{2}}\nabla u_\varepsilon)+\varepsilon u_\varepsilon=(m-\Phi[u_\varepsilon]u_\varepsilon^+,\qquad (x,t)\in Q_T,\qquad (2.4)$$

$$\frac{\partial u_\varepsilon}{\partial t}=0,\qquad (2.3)$$

$$u_\varepsilon(x,0)=u_\varepsilon(x,T)\qquad (2.4)$$

Where $s^+=\max\{0,s\}$ and $A(u_\varepsilon)=mu_\varepsilon^{m-1}+\varepsilon,\varepsilon$ is a sufficiently small positive constant , The desired solution will be obtained as the limit point of the solutions of the problem (1.1)-(1.3). In the following, we introduce a map by the following problem

$$\frac{\partial u_\varepsilon}{\partial t}-div((|A(u_\varepsilon)\nabla u_\varepsilon|^2+\varepsilon)^{\frac{p-2}{2}}\nabla u_\varepsilon)+\varepsilon u_\varepsilon=f,\qquad (x,t)\in Q_T,\qquad (2.5)$$

$$\frac{\partial u_\varepsilon}{\partial t} = 0, \qquad (x,t) \in \partial\Omega \times (0,T), \qquad (2.6)$$

$$u_\varepsilon(x,0) = u_\varepsilon(x,T), \qquad x \in \Omega, \qquad (2.7)$$

Then we can define a map $u_\varepsilon = Gf$ with $G: C_T(\overline{Q}_T) \to C_T(\overline{Q}_T)$. by applying classical estimated (see [9]), we can know that $\|u_\varepsilon\|_{L^\infty(Q_T)}$ is bounded by $\|f\|_{L^\infty(Q_T)}$ and u_ε is Holder continuous in Q_T . Then by the Arzela-Ascoli theorem, the map G is compact. So the map is a compact continuous map. Let $f(u) = (m - \Phi[u_\varepsilon]u_\varepsilon^+)$ where $u_\varepsilon^+ = \max\{u_\varepsilon, o\}$ we can see that the nonnegative solution of problem (1.1)-(1.3) is also a nonnegative solution solves $u_\varepsilon = G(m - \Phi[u_\varepsilon]u_\varepsilon^+)$. So we will study the existence of the nonnegative fixed points of the map $u_\varepsilon = G((m - \Phi[u_\varepsilon])u_\varepsilon^+)$ instead of the nonnegative solution of problem (1.1)-(1.3).

3. Proof of the main results :First, by the same way as in [5], we can get the non-negativity of the solution of problem (2.2)-(2.4) .

Lemma 1 *If* a nontrivial function $u_\varepsilon \in C(\overline{Q}_T)$ solves $u_\varepsilon = G((m - \Phi[u_\varepsilon])u_\varepsilon^+)$, then

$$u_\varepsilon(x,t) \geq 0 \qquad \forall x,t \in \overline{Q}_T$$

In the following, by the Moser iterative technique, we will show the priori estimate for the upper bound of nonnegative periodic solution of problem (2.5)-(2.7). Here and below we denote by $\|.\|_p$ $(1 \leq p \leq \infty)$ then $L^p(\Omega)$ norm.

Lemma 2 Let $u_\varepsilon(x,t)$ be a nontrivial periodic solution which solves $u_\varepsilon = T(1, \sigma f(u_\varepsilon))$, $\sigma \in [0,1]$ and then there exists a positive constant K independent of σ and ε , such that

$$\|u_\varepsilon\| < k, \qquad (3.1)$$

Where $u_\varepsilon(t) = u_\varepsilon(.,t)$.

Proof: suppose u_ε is a nontrivial periodic solution, Multiplying Equation (2.5) by u_ε^s where $(s \geq 0)$ and integrating over Ω , we get

$$\frac{1}{s+1}\frac{d}{dt}\|u_\varepsilon(t)\|_{s+1}^{s+1} + \frac{sp^p m^{p-1}}{[m(p-2)+s+1]^p}\left\|\nabla(u_\varepsilon^{\frac{m(p-2)+s+1}{p}}(t))\right\|_p^p \leq \|m(x,t)\|_{L^\infty(\Omega\times(0,T))}\|u_\varepsilon(t)\|_{s+1}^{s+1}$$

Where $(a(x,t) - \Phi[u_\varepsilon] \leq Mu_\varepsilon)$ and $M = \sup_{(x,t)} a(x,t) \in \overline{Q}_T$

$$\frac{d}{dt}\|u_\varepsilon(t)\|_{s+1}^{s+1} + \frac{sp^p m^{p-1}}{[m(p-2)+s+1]^p}\left\|\nabla(u_\varepsilon^{\frac{m(p-2)+s+1}{p}}(t))\right\|_p^p \leq M(s+1)\|u_\varepsilon(t)\|_{s+1}^{s+1} \qquad (3.2)$$

And hence:

$$\frac{d}{dt}\|u_\varepsilon(t)\|_{s+1}^{s+1} + C\left\|\nabla(u_\varepsilon^{\frac{m(p-2)+s+1}{p}}(t))\right\|_p^p \leq C(s+1)\|u_\varepsilon(t)\|_{s+1}^{s+1}, \qquad (3.3)$$

Where C is a positive constants independent of u_ε, k and m .

Assume that $\|u_\varepsilon(t)\|_\infty \neq 0$ and set

$$s_k = p^k + m - \frac{p}{p-2}, \quad \alpha_k = \frac{p(s_k+1)}{m(p-2)+s_k+1}, \quad u_k(t) = u_\varepsilon^{\frac{m(p-2)+s+1}{p}}(t) \text{ where } (k = 0,1,\ldots)$$

Then $\alpha_k < p,\ m_k = p^{k-1} + m_{k-1}$.

For convenience, we denote by C a positive constant independent of u_ε, k and m , which may take different values .

From (3.3) we obtain

$$\frac{d}{dt}\|u_k(t)\|_{\alpha_k}^{\alpha_k} + C\|\nabla u_k(t)\|_p^p \le C(s+1)\|u_k(t)\|_{\alpha_k}^{\alpha_k}, \tag{3.4}$$

We can using the Gagliardo-Nirenberg inequality, we have

$$\|u_k(t)\|_{\alpha_k} \le D\|\nabla u_k(t)\|_p^{\theta}\|u_k(t)\|_1^{1-\theta} \tag{3.5}$$

With

$$\theta_k = \frac{(p-1)m_k + p}{m_k + 2}\frac{N}{(p-1)N+2} \in (0,1).$$

By inequalities (3.4)-(3.5) and the fact that $\|u_k(t)\|_1 = \|u_{k-1}(t)\|_{\alpha_{k-1}}^{\alpha_{k-1}}$, we obtain the following differential inequality:

$$\frac{d}{dt}\|u_k(t)\|_{\alpha_k}^{\alpha_k} \le -C\|u_k(t)\|_{\alpha_k}^{\frac{p}{\theta}}\|u_k(t)\|_1^{\frac{p(\theta-1)}{\theta}} + C(s_k+1)\|u_k(t)\|_{\alpha_k}^{\alpha_k}$$

$$\le -C\|u_k(t)\|_{\alpha_k}^{\frac{p}{\theta}}\|u_{k-1}(t)\|_{\alpha_{k-1}}^{\frac{(\theta-1)}{\theta}\alpha_{k-1p}} + C(s_k+1)\|u_k(t)\|_{\alpha_k}^{\alpha_k}.$$

Let

$$\xi_k = \max\{1, \sup_t \|u_k(t)\|_{\alpha_k}\},$$

We have

$$\frac{d}{dt}\|u_k(t)\|_{\alpha_k}^{\alpha_k} \le \|u_k(t)\|_{\alpha_k}^{\frac{\alpha_k(m_k+1)}{m_k+2}}\{-C\|u_k(t)\|_{\alpha_k}^{\frac{p}{\theta}-\frac{\alpha_k(m_k+1)}{m_k+2}}\xi_{k-1}^{\frac{(\theta-1)}{\theta}\alpha_{k-1p}} + C(s_k+1)\|u_k(t)\|_{\alpha_k}^{\frac{\alpha_k}{m_k+2}}\} \tag{3.6}$$

By young's inequality

$$\alpha b \le \varepsilon\alpha^{p'} + \varepsilon^{-\frac{q'}{p'}}b^{q'},$$

Where $p' > 1, q' > 1, \alpha > 0, b > 0, \varepsilon > 0$ and $\frac{1}{p'} + \frac{1}{q'} = 1.$ set

$$\alpha = \|u_k(t)\|_{\alpha_k}^{\frac{\alpha_k}{m_k+2}}, \qquad b = s_k + 1, \qquad \varepsilon = \frac{1}{2}\xi_{k-1}^{\frac{(\theta-1)}{\theta}\alpha_{k-1}p},$$

$$p' = l_k = \frac{p(s_k+1)}{\alpha_k\theta} - s_k - 2 = \frac{(s_k+1)(s_k+p)(p-1)N+2}{N((p-1)s_k+p)} - s_k - 2,$$

Then we obtain

$$(s_k+1)\|u_k(t)\|_{\alpha_k}^{\frac{\alpha_k}{m_k+2}} \le \frac{1}{2}\|u_k(t)\|_{\alpha_k}^{\frac{p}{\theta}-\frac{\alpha_k(m_k+1)}{m_k+2}}\xi_{k-1}^{\frac{(\theta-1)}{\theta}\alpha_{k-1p}} + C(s_k+1)^{\frac{l_k}{l_k-1}}\xi_{k-1}^{\frac{(1-\theta)}{\theta}\alpha_{k-1p}\frac{1}{l_k-1}} \tag{3.7}$$

Here we have used the fact that $p' = l_k > r > 1$ for some r independent of k. in fact, it is easy to verify that $\lim_{k\to\infty} l_k = +\infty.$

Donate

$$\alpha_k = \frac{(p-1)l_k}{l_k-1}, \qquad b_k = \frac{1-\theta}{\theta}\frac{p\alpha_{k-1}}{l_k-1},$$

And combining (3.7) with (3.6) we have

$$\frac{d}{dt}\|u_k(t)\|^{\alpha_k} \le \|u_k(t)\|_{\alpha_k}^{\frac{\alpha_k(m_k+1)}{m_k+2}}\{\frac{-C}{2}\|u_k(t)\|_{\alpha_k}^{\frac{\alpha_k}{\theta}-\frac{\alpha_k(m_k+1)}{m_k+2}}\xi_{k-1}^{\frac{(\theta-1)}{\theta}\alpha_{k-1p}} + C(s_k+1)^{\alpha_k}\xi_{k-1}^{b_k}\}^{\alpha_k}. \tag{3.8}$$

Then

$$(m_k+2)\frac{d}{dt}\|u_k(t)\|_{\alpha_k}^{\frac{\alpha_k}{m_k+2}} \le -C\|u_k(t)\|_{\alpha_k}^{\frac{\alpha_k}{\theta}-\frac{\alpha_k(m_k+1)}{m_k+2}} \xi_{k-1}^{\frac{(\theta-1)}{\theta}\alpha_{k-1}p} + C(m_k+2)^{\alpha_k}\xi_{k-1}^{b_K}. \qquad (3.9)$$

From the periodicity of $u_k(t)$, we know that there exists t_0 at which $\|u_k(t)\|_{\alpha_k}$ reaches its maximum and thus the left hand of (3.9) vanishes. Then we obtain

$$\|u_k(t)\|_{\alpha_k} \le \{C[(m_k+2)^{\alpha_k}\xi_{k-1}^{\frac{(\theta-1)}{\theta}\alpha_{k-1}p}]\}^{\frac{1}{\alpha_k}},$$

Where

$$\alpha_k = \frac{p}{\theta} - \frac{\alpha_k(m_k+1)}{m_k+2} = \frac{\alpha_k l_k}{m_k+2}.$$

Therefore we conclude that

$$\|u_k(t)\|_{\alpha_k} \le \{C(m_k+2)^{\alpha_k}\xi_{k-1}^{b_k+\frac{(\theta-1)}{\theta}\alpha_{k-1}p}\}^{\frac{1}{\alpha_k}} = \{C(m_k+2)^{\alpha_k}\}^{\frac{m_k+2}{\alpha_k l_k}}\zeta_{k-1}^{\frac{(1-\theta)(m_k+2)\alpha_{k-1}p}{(l_k-1)^{\theta}}}$$

Since $\frac{m_k+2}{(l_k-1)^{\theta}} = \frac{\alpha_k}{1-\theta\alpha_k}$, $\frac{m_k+2}{\alpha_k l_k}$ and α_k are bounded, we get

$$\|u_k(t)\|_{\alpha_k} \le Cp^{k\alpha'}\xi_{k-1}^{\frac{(1-\theta)\alpha_{k-1}p}{(p-\theta\alpha_k)}},$$

Where $\alpha'>1$ is a positive constant independent of k, as $\alpha_k = \frac{p(m_k+2)}{m_k+p} < p$ implies that

$\frac{(1-\theta)\alpha_{k-1p}}{(p-\theta\alpha_k)} \le \frac{(1-\theta)\alpha_{k-1p}}{(p-\theta p)} \le p$ and $\xi_{k-1} \ge 1,$ then we have

$$\|u_k(t)\|_{\alpha_k} \le CA^k\xi_{k-1}^p$$

Or

$$\ln\|u_k(t)\|_{\alpha_k} \le \ln\xi_k \le \ln C + k\ln A + p\ln\xi_{k-1},$$

Where $A = p^{\alpha'} > 1$. Thus

$$\ln\|u_k(t)\|_{\alpha_k} \le \ln C\sum_{i=0}^{k-2}p^i + p^{k-1}\ln\xi + \ln A(\sum_{j=0}^{k-2}(k-j)p^j)$$
$$\le (p^{k-1}-1)\ln C + p^{k-1}\ln\xi + f(k)\ln A,$$

Or

$$\|u_k(t)\|_{m_k+2} \le \{C^{\frac{p^{k-1}-1}{p-1}}\xi^{p^{k-1}}Af(k)\}^{\frac{p}{m_k+p}}$$

Where

$$f(k) = \frac{k-p(k+1)-p^{k-1}+2p^k}{(p-1)^2}.$$

Letting $k\to\infty$, we obtain

$$\|u(t)\|_{\infty} \le C\xi^{p-1} \le C(\max\{1,\sup_t\|u(t)\|_2\})^{p-1}. \qquad (3.10)$$

On the other hand, it following from (3.3) with m=0 that

$$\frac{d}{dt}\|u(t)\|_2^2 + C_1\|\nabla u(t)\|_p^p \le C_2\|u(t)\|_2^2 \qquad (3.11)$$

By Holder's inequality and sobolev's theorem, we have

$$\|u(t)\|_2 \le |\Omega|^{\frac{1}{2}-\frac{1}{p}} \|u(t)\|_p \le C\,|\Omega|^{\frac{1}{2}-\frac{1}{p}} \|\nabla u(t)\|_p \qquad (3.12)$$

Combined with (3.11) , it yields

$$\frac{d}{dt}\|u(t)\|_2^2 + C_1\|\nabla u(t)\|_2^p \le C_2\|u(t)\|_2^2. \qquad (3.13)$$

By young's inequality, it follows that

$$\frac{d}{dt}\|u(t)\|_2^2 + C_1\|\nabla u(t)\|_2^p \le C_2 \qquad (3.14)$$

Where $C_i\ (i=1,2)$ are constant independent of u. Taking the periodicity of u into account, we infer from (3.14) that $\|u(t)\|_2 \le C$.

Which together with (3.10) implies (3.1). The proof is completed.

Corollary 1 There exists a positive constant R independent of ε, such that

$$\deg(I - G(1,(m-\Phi[u_\varepsilon])u_\varepsilon^+), B_R, 0) = 1,$$

Where B_R is a ball centered at the origin with radius R in $L^\infty(Q_T)$.

Proof it follows from Lemma 2 that there exists appositive constant R independent of ε, such that

$$u_\varepsilon \ne G(\sigma(m-\Phi[u_\varepsilon]u_\varepsilon^+), \qquad \forall u_\varepsilon \in \partial B_R, \qquad \sigma\in[0,1].$$

So the degree is will defined on B_R . from the homotopy invariance of the Leray-schauder degree and the existence and uniqueness of the solution of G(1,0) , we can see that

$$\deg(1-G((m-\Phi[u_\varepsilon]u_\varepsilon^+), B_R, 0) = \deg(1-G(1,\sigma(m-\Phi[u_\varepsilon]u_\varepsilon^+), B_R, 0)$$
$$= \deg(1-G(1,0), B_R, 0) \;\; = 1.$$

The proof is completed.

Lemma 3 There exist a constants $r>0$ and $\varepsilon>0,$ such that no non-trivial solution u_ε of the equation, $G((m-\Phi[u_\varepsilon]u_\varepsilon^+)$ satisfy

$$0 < \| u_\varepsilon\|_{L^\infty(Q_T)} \le r,$$

Proof By contradiction , let u_ε be a non-trivial solution of $u_\varepsilon = G((m-\Phi[u_\varepsilon]\, u_\varepsilon^+)$ satisfying $0<\| u_\varepsilon\|_{L^\infty(Q_T)} \le r,$

For any given $\phi(x)\in C_0^\infty(\Omega),$ multiplying (2.5) by $\frac{\phi^2}{u_\varepsilon}$ and integrating over $Q_T^* = B_\delta(x_0)\times(0,T),$ we obtain:

$$\iint_{Q_T^*} \frac{\phi^2}{u_\varepsilon}\frac{\partial u_\varepsilon}{\partial t}dtdx + \iint_{Q_T^*}((|B(u_\varepsilon)\nabla u_\varepsilon|^2+\varepsilon)^{\frac{p-2}{2}} B(u_\varepsilon)\nabla u_\varepsilon \nabla(\frac{\phi^2}{u_\varepsilon}))dtdx \qquad (3.15)$$
$$\le \iint_{Q_T^*} \phi_1^2(m-\varepsilon-\Phi[u])dtdx.$$

Due to the periodicity of u_ε with respect t we have

$$\iint_{Q_T^*} \frac{\phi^2}{u_\varepsilon}\frac{\partial u_\varepsilon}{\partial t}dtdx = \int_\Omega \phi^2 \int_o^T \frac{\partial(\ln u_\varepsilon)}{\partial t}dtdx = 0. \qquad (3.16)$$

The second term on the left –hand side in (3.15) can be rewritten as

$$\iint_{Q_T^*}((|B(u_\varepsilon)\nabla u_\varepsilon|^2+\varepsilon)^{\frac{p-2}{2}} B(u_\varepsilon)\nabla u_\varepsilon \nabla(\frac{\phi^2}{u_\varepsilon})dtdx$$

$$=\iint_{Q_T^*}((|B(u_\varepsilon)\nabla u_\varepsilon|^2+\varepsilon)^{\frac{p-2}{2}}B(u_\varepsilon)\nabla u_\varepsilon\nabla(\phi.\frac{\phi}{u_\varepsilon}))dtdx$$

$$=\iint_{Q_T^*}((|B(u_\varepsilon)\nabla u_\varepsilon|^2+\varepsilon)^{\frac{p-2}{2}}B(u_\varepsilon)\nabla u_\varepsilon(\frac{\phi}{u_\varepsilon}\nabla\phi+\nabla(\frac{\phi}{u_\varepsilon})\phi))dtdx$$

$$=\iint_{Q_T^*}((|B(u_\varepsilon)\nabla u_\varepsilon|^2+\varepsilon)^{\frac{p-2}{2}}B(u_\varepsilon)\nabla(\frac{\phi}{u_\varepsilon})(u_\varepsilon\nabla\phi-u_\varepsilon^2\nabla(\frac{\phi}{u_\varepsilon}))dtdx$$

$$+\iint_{Q_T^*}((|B(u_\varepsilon)\nabla u_\varepsilon|^2+\varepsilon)^{\frac{p-2}{2}}B(u_\varepsilon)\nabla u_\varepsilon\nabla(\frac{\phi}{u_\varepsilon})dtdx \qquad (3.17)$$

$$=\iint_{Q_T^*}((|B(u_\varepsilon)\nabla u_\varepsilon|^2+\varepsilon)^{\frac{p-2}{2}}B(u_\varepsilon)|\nabla\phi|^2dtdx$$

$$-\iint_{Q_T^*}((|B(u_\varepsilon)\nabla u_\varepsilon|^2+\varepsilon)^{\frac{p-2}{2}}B(u_\varepsilon)(u_\varepsilon\nabla\phi-u_\varepsilon\nabla\phi)\nabla(\frac{\phi}{u_\varepsilon})dtdx$$

$$=\iint_{Q_T^*}((|B(u_\varepsilon)\nabla u_\varepsilon|^2+\varepsilon)^{\frac{p-2}{2}}B(u_\varepsilon)|\nabla\phi|^2dtdx$$

$$-\iint_{Q_T^*}((|B(u_\varepsilon)\nabla u_\varepsilon|^2+\varepsilon)^{\frac{p-2}{2}}B(u_\varepsilon)u_\varepsilon^2\left|\nabla(\frac{\phi}{u_\varepsilon})\right|^2dtdx$$

Thus:

$$\iint_{Q_T^*}((|B(u_\varepsilon)\nabla u_\varepsilon|^2+\varepsilon)^{\frac{p-2}{2}}B(u_\varepsilon)\nabla u_\varepsilon\nabla(\frac{\phi^2}{u_\varepsilon}))dtdx \qquad (3.18)$$

$$\leq\iint_{Q_T^*}((|B(u_\varepsilon)\nabla u_\varepsilon|^2+\varepsilon)^{\frac{p-2}{2}}B(u_\varepsilon)|\nabla\phi|^2dtdx$$

Combining (3.16) with (3.15)(3.18) , we obtain

$$\iint_{Q_T}\phi_1^2(m-\varepsilon-\Phi[u_\varepsilon])dtdx\leq\iint_{Q_t^*}((|B(u_\varepsilon)\nabla u_\varepsilon|^2+\varepsilon)^{\frac{p-2}{2}}B(u_\varepsilon)|\nabla\phi|^2dtdx \qquad (3.19)$$

Let μ_1 be the first eigenvalue of the p-Laplacian equation on Ω with zero boundary condition and $\phi_1(x)$ be the corresponding eigenfunction, we have:

$$\int_\Omega|\nabla\phi_1|^p\,dx=\mu_1\int_\Omega|\phi_1|^p\,dx \qquad (3.20)$$

From theorem 5.1 and also some remarks in [[10].pp.238, 243], it follows that there exists a constant $\gamma=\gamma(N,P)$ such that

$$\sup_{[(x_0,t_0)+Q(\frac{1}{2}r_0,\frac{1}{2}p)]}|B(u_\varepsilon)\nabla u_\varepsilon|=C(N,p,r_0,a_0,\mu_1)\left(\iint_{[(x_0,t_0)+Q(r_0,p)]}|B(u_\varepsilon)\nabla u_\varepsilon|^p\,dtdx\right)^{\frac{1}{2}}\wedge\frac{1}{2}\left(\frac{a_0}{4\mu_1}\right)^{\frac{1}{2-p}}$$

For any $(x_0,t_0)\in Q_{(T,3T)}=\Omega\times(T,3T),[(x_0,t_0)+Q(r_0,p)]$ and $p=\min\left\{T,\frac{\sqrt{\alpha_0 r_0}}{2^{\frac{p+6}{2}}}\right\}$ On the other hand, by (2.2) with (2.4) , we have

$$\iint_{Q_T} |B(u_\varepsilon)\nabla u_\varepsilon|^p\, dtdx \le \max_{Q_T} |\alpha(x,t)| \iint_{Q_T} (|u|^{m+1} + |u|^2) dtdx .$$

So

$$\sup_{[(x_0,t_0)+Q(\frac{1}{2}r_0,\frac{1}{2}p)]} |B(u_\varepsilon)\nabla u_\varepsilon| = C(N,p,r_0,a_0,\mu_1) \iint_{Q_T} (|u|^{m+1} + |u|^2) dtdx \wedge \frac{1}{2}\left(\frac{a_0}{4\mu_1}\right)^{\frac{1}{2-p}}$$

Which implies

$$\|B(u_\varepsilon)\nabla u_\varepsilon\|_{L^\infty(B(x_0,r_0)\times(0,T))} \le C(\|u\|^{\frac{m+1}{2}}_{L^\infty(Q_T)} + \|u\|_{L^\infty(Q_T)}) \wedge \frac{1}{2}(\frac{a_0}{4\mu_1})^{\frac{1}{2-p}}$$

Where C is a constant independent of ε, from $\varepsilon \in (0,\frac{1}{2})$ we have $B(\varepsilon) = mu_\varepsilon^{m-1} + \varepsilon \le mr^{m-1} + \frac{1}{2}$ By the approximating process, we can let $\phi = \phi_1$ is the positive eigenfunction of the first eigenvalue μ_1 ,then we

$$\iint_{B(x_0,\frac{1}{2}r_o)\times(0,T)} \phi_1^2 (m - \varepsilon - \Phi[u_\varepsilon])dtdx$$

$$\le \iint_{B(x_0,\frac{1}{2}r_o)\times(0,T)} ((|B(u_\varepsilon)\nabla u_\varepsilon|^{p-2} + \varepsilon^{\frac{p-2}{2}})B(u_\varepsilon)|\nabla\phi|^2 dtdx$$

$$\le \iint_{B(x_0,\frac{1}{2}r_o)\times(0,T)} (C(r^{\frac{m+1}{2}} + r)^{p-2} \wedge \frac{a_0}{4\mu_1} + \varepsilon^{\frac{p-2}{2}})(mr^{m-1} + \frac{1}{2}) \int_{B_\delta(x_0)} \phi_1^2 dx. \qquad (3.21)$$

On the other hand

$$\iint_\Omega \phi_1^2 (a - \varepsilon - \Phi[u])dtdx$$

$$\ge \iint_\Omega \phi_1^2 (m - \varepsilon - k\,\|u\|^2_{L^2})dtdx \qquad (3.22)$$

$$\ge \int_{B_\delta(x_0)} \phi_1^2 \int_0^T (m - \varepsilon - k\,\|u\|^2_{L^2})dtdx$$

$$\ge T\,(m_0 - \varepsilon - kr^2 |\Omega| \int_{B_\delta(x_o)} \phi_1^2 dx .$$

Where Ω denotes the Lebesgue measure of the domain Ω , and then we obtain

$$m_0 \le \varepsilon + kr^2|\Omega| + (C_{\mu_1}(r^{\frac{m+1}{2}} + r)^{p-2} \wedge \frac{a_0}{4} + \mu_1\varepsilon^{\frac{p-2}{2}})(mr^{m-1} + \frac{1}{2}). \qquad (3.23)$$

Obviously if we let

$$r \le \min\left\{ \sqrt[m-1]{\frac{1}{2m}}, (\frac{m_0}{4k})^{\frac{1}{2}}, \frac{1}{2}(\frac{m_0}{4C\mu_1})^{\frac{1}{p-2}}, 1 \right\} \qquad (3.24)$$

We can get

$$\alpha_0 \le \frac{\alpha_0}{4} + (\frac{\alpha_0}{4} \wedge \frac{\alpha_0}{4} + \frac{\alpha_0}{4}) = \frac{3\alpha_0}{4}.$$

This inequality does not hold. Therefore there exists one positive constant r > 0, such that no nontrivial solution u_ε of the equation $G((m - \Phi[u_\varepsilon])u_\varepsilon^+)$ satisfy $0 < \| u_\varepsilon\|_{L^\infty(Q_T)} \le r$

Thus we complete the proof.

Corollary 2 There exists a small positive constant r which is independent of ε and satisfies $r < R$ such that

$$\deg(I - G(1,(m - \Phi[u_\varepsilon])u_\varepsilon^+, B_r, 0) = 0,$$

Where B_r is a ball centered at the origin with radius r in $L^\infty(Q_T)$.

Proof same way for lemma 3, we can see that there exists a positive constant $o < r < R$ independent of ε, such that:

$$u_\varepsilon \neq G(\tau,(m-\Phi[u_\varepsilon])\ u_\varepsilon^+ + 1 - \lambda), \forall u_\varepsilon \in \partial B_r, \lambda \in [0,1].$$

Thus the degree is well defined on B_r , By Lemma 3, we can easy to infer that $u = G(0,(m - \Phi[u]u^+)$ admits no solution in B_r , Then by homotopic invariance of the Leray-schauder degree , we get

$$\deg(I - G(1,(m-\Phi[u_\varepsilon]\ u_\varepsilon^+), B_r, 0) = \deg(1 - G(0,(m-\Phi[u_\varepsilon]\ u_\varepsilon^+ + 1), B_r, 0) = 0.$$

The proof is completed.

Now we show the proof of the main result of this paper.

Theorem 1 if assumption (B1),(B2) hold then the problem (1.1)-(1.3) admits a nontrivial nonnegative periodic solution u_ε .

Proof Using corollaries 1 and 2 , we have

$$\deg(1 - G(f\ (.)), \Gamma, 0) = 1,$$

Where $\Gamma = B_R \setminus B_r, B_\mu$ is a ball centered at the origin with radius $\mu \in L^\infty(Q_T)$, R and r are positive constants and R > r . By the theory of the Leray-Schauder degree and Lemma 1, we can conclude that problem (2.2)-(2.4) admits a nontrivial nonnegative periodic solution u_ε By Lemma 3 and a similar method to that in [11], we can obtain

$$\|\nabla u_\varepsilon\|_{L^p(Q_T)} \leq C, \qquad \left\|\frac{\partial u_\varepsilon}{\partial t}\right\| \leq C$$

Combining with the regularity results [10] a similar argument to that in [11], we can prove that the limit function of is nonnegative nontrivial periodic solution of problem (1.1)-(1.3).

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الحل الدوري لصنف معادلة القطع المكافئ ذات الاضمحلال المضاعف مع شروط نيوتن الحدودية

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الخلاصة:

لقد تم في هذا البحث ، دراسة الحل الدوري لصنف من معادلات القطع المكافئ ذات الاضمحلال المضاعف مع شروط انيومن الحدودية وباستخدام نظرية Leray–Schauder degree ولقد حصلنا على وجود للحل الدوري غير التافه.