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ON FUZZY-⊕-S-MODULES

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A R T I C L E I N F O

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A B S T R A C T

A module M is a ⊕-Supplemented if M=⊕N_i such that N_i is a direct summand of M. In this paper we introduce the notion of Fuzzy ⊕-Supplemented Module (Fuz-⊕-S-Mod) as a generalization of ⊕-supplemented module. We investigate several characterizations and properties of this concept. Moreover we investigated several relationships among Fuz-⊕-S-Mod and other Fuzzy concepts such as fuzzy D_1-module (Fuz-D_1-module), Fuzzy Lifting Module (Fuz-L-mod) and Amply Fuzzy Supplemented module (A-Fuz-S-mod) are studied.

1. Introduction

Fuzzy sets (Fuz-S) introduced by [12], (Zadeh, 1965) showed meaningful applications in many fields of study such as mathematics, biology, engineering and almost of scinces. Let X be a nonempty set. A fuzzy set A drawn from X is defined as $A=x,\mu A(x):x\in X$, where $\mu A(x)$: $X\longrightarrow[0,1]$ is the membership function of the fuzzy set A. "The notion of fuzzy subsets of a set S $\neq\emptyset$ as a function from S into [0,1] was first developed by Zadeh [12]". "The concepts of fuzzy module (Fuz-mod) and fuzzy submodule (Fuz-sub) was introduced by Negoita and Ralescu in [7]". "Clark, Lomp, Vanaja and Wisbauer [3], defined small submodule of M and lies above a direct summand of M". "In [4], Idelhadj and Tribak defined ⊕-supplemented module (⊕-S-mod) as a submodule of a module M has supplement that is a direct summand of M". "Alkan and Harmanci [2], introduced D_1-mod". "In [8], Orhan, Tutuncu and Tribak studied a lifting module (L-mod) in detailes". " In [10], Top studied a linearly compact of any module M". " In [6], Michielsen defined simple and semisimple modules (Sem-mod)".

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In this paper we give the definition of Fuz-⊕-S-Module (Fuz-⊕-S-mod) and some his relationships with other Fuzy modules. In Theorem 3.10, "we wiill prove that any Fuz-submodule lies above a direct summand (⊕-Fuz-sub-Li-above) of amply fuzzy supplemented module (A-Fuz-S-mod) M is a Fuz-⊕-S-Module". Also we will prove that if a submodule N is a ⊕-Fuz-sub-Li-above and N=⊕ Fuz-mod M has Fuz-Lin-compact, then M is a Fuz-⊕-S-module (see Theorem 3.13).

2. Preliminaries

First we introduce some concepts and results of Fuz-sets and Fuz-submodules, which we need to develop our paper. We can obtain the definition of module by doing mapping between ring and commutative group:

Definition 2.1. (see [3]). If we have μ: R×M →M, subject to the conditions that for all a; b∈R and c; d∈M ∋ R is a ring and M is a comm. group, then M is called R-module if:

(1) (a; c + d) = (a; d) + (a; c);
(2) (a + s; d) = (a; d) + (b; d);
(3) (ab; d) = (a; (b; d));
(4) if 1 ∈ R, then (1; d) = d.

Definition 2.2. (see [10]) "For R-module M, a maximal submodule A of M is a sub-module M≠A for which any other submodule N, if A⊆N⊆M, then N=M or N=A."
"Equivalently; A is a maximal submodule if and only if the quotient module M/A is a simple module".

Definition 2.3. (see [11]) "Let M be an R-module. A fuzzy set X of M is called fuzzy module of an R-module M (Fuz-mod) if":
(1) η(a-b) ≥ min{ η (a), η (b)} for all a, b ∈ M;
(2) η (ra) ≥ η (a), for all a ∈ M and r ∈ R;
(3) η (0)=1.

Definition 2.4. (see [11]) "A Fuz-sub of a module M is a Fuz-subset μ∈ $[0, 1]^M$ such that":
(1) μ(θ)=1;
(2) μ(x+y) ≥μ(x)μ(y) ∀ x, y ∈ M and
(3) μ(ra)≥μ(a), for all r∈R, a∈M.

Remark 2.5. We denote to the all Fuzzy submodules (Fuz-subs) of M by F(M).

Definition 2.6. (see [9]). Let μ and σ∈F(M). Then we call μ a Fuz-supplement of σ, if μ is minimal with respect to μ+σ=χM such that χM is the sum of Fuzzy supplement and minimal Fuzzy". "Equivalently, μis called a Fuz-supplement of σ if and only if μ+σ=χM and μ∩σ $\ll_f$ μ". "A Fuz-sub μ of M is called a Fuz-supplement, if there is a Fuz-sub σ of M such that μ is a Fuz-supplement of σ". "Therefore any R-module M (or χM) is called Fuz-supplemented, if every Fuz-sub has a Fuz-supplement in M".

Definition 2.7. (see [9]) "If for every v ∈ F(M) with μ+v =χM there is a supplement σ of μ such that σ ⊆v, then it is said that μ has A-Fuz-supplement in M".

Definition 2.8. (see [9]) "A module M (or χM) is called A-Fuz- supplemented if any Fuz-sub of M has a A-Fuz-supplement in M". "Thus every A-Fuz-S-mod is F-supplemented".

Definition 2.9. (see [9]) "Let M be an R-module and μ ∈ F(M). Then μ is said to be a Fuz-small sub of M or (χM), if for any v ∈ F(M), μ+v = χM implies that v= χM". (μ$\ll_f$M or μ $\ll_f\chi$M).

3. Main Results

In this section, we introduce and determine the necessary conditions in order to obtain the second generalization of ⊕-supplemented module namely Fuz-⊕-S-mod. First we give the following definition.

Definition 3.1. If N is a F-sub-module of a F-module M such that N has Fuz-supplement that is a Fuz-⊕-mod, then M is called Fuz-⊕-S-mod.

Now we need to define cosets; if G is a group, and H is a subgroup of G, and g is an element of G, then gH ={gh : h an element of H } is the left coset of H in G with respect to g.

Definition 3.2. Let M be a Fuz-mod. Then M is Fuz-Lin-compact if for the family of Fuz-cosets x_i∈M and Fuz-subs M_i ⊆M ∋ ∩x_i with collection of M_i is a non empty set. Then the fuzzy of this intersection is not empty.

Definition 3.3. Any module M is Fuz-local if Rad(M) is a Fuz-small max-sub of M.

Lemma 3.3. Any Fuz-sub- Lin- compact N of a Fuz-mod M has Fuz-supplement in M.

Proof. If R and S are Fuz-subs of a Fuz-mod M, then μ(x)≤ η(x) ∋ A and S subsets of M. Assum that N is a Fuz-Lin-compact of M and M=R+S. If ηM and S_1 subset of S such that S_1=S, then R+S_1=M. But S∈ηM, so ηM≠0. Suppose that S_2 is a chain in ηM. Hence we take an inverse family of Fuz-subs of S_2. Now lower of (S_2) = fuzzy of intersection of S_2= min(S_2). By Definition. 3.2,
R with fuzzy of (intersection of S_2)≡ minimum of S_2
≡ fuzzy of intersection of (R+S_2)
≡ minimum of (R+S_2)
≡ M.
So fuzzy intersection of S_2 in ηM. Therefore K is a Fuz-submodule of M and Fuz-minimal element belong to ηM such that M=R+K. Then K is a Fuz-supplement of R and K is a subset of S. Thus, R has A-Fuz-supplement belong to M.

Now we introduce a new definitions which we use it later.

Definition 3.4. If for all A, B are Fuz-sub of M and C⊆A, A∩C=C and C is Fuz-sub of A, a decomposition M = C⊕B and A∩C is F-small in M, then M is called Fuz-L-mod.

Definition 3.5. We say M is Fuz-D_1-mod if A is a Fuz-sub of Fuz-mod M and M=M_1⊕M_2 ∋ M_1 submodule of A and M_2∩A$\ll_F$ M_2.

Lemma 3.6. If R is Fuz-ring and M is Fuz-D_1-mod over R, then M is a Fuz-L-mod.

Proof. Let M be a Fuz-D_1-mod. Every F-sub K of M we have decomposition of M from A_1 and A_2 ∋ A_1 ≤K and A_2∩K ≪A_2. By Definition. 3.4, we obtain the result.

Definition 3.7. A module M is called Fuz-Sem-simple if M=⊕ Fuz-simple subs. Or, if every Fuz-sub M=⊕M. Equivalently, for any Fuz-ring R, a left Fuz-mod M is called Fuz-Sem-simple-mod if every fuzzy short exact sequence 0→M_1→M→M_2→0 of Fuz-mods splits.

"A module A is called fuzzy injective (Fuz-injective) if A is a Fuz-sub of some other left fuzzy *R*-module *M*, then there exists another Fuz-sub B of *M* such that *M* is the fuzzy of A+B, i.e. A+*B*=M and A∩B={0}". From [7], "let R be a ring. If any simple module over R is injective, then R is called V-ring". So, we can introduce a meaning of the fuzzy V-ring (Fuz-V-ring); if any Fuz-simple mod over R is a Fuz-injective, then R is called Fuz-V-ring.

Lemma 3.8. If R is a left Fuz-Sem-simple V-ring, then M is Fuz-⊕-S-mod.

Proof. Since R is a Fuz-V-ring, then any Fuz-simple mod over R is a Fuz-injective. We have R is Fuz-Sem-simple ring. Thus M is a Fuz-⊕-S- module.

Definition 3.9. If M is a F-R-mod and K ≤U ≤M s.t U/K small in M/K, then we say U Fuz-Lie-above K. We know that, U is Fuz-Li-above a submodule K of M iff K≤ U and for every Fuz-sub T of M and M equal the sum of U and T, so is Fuz-D_1-module.

Theorem 3.10. Let M be an A-Fuz-S-mod. If every Fuz-sub of ⊕-Fuz-sub-Li-above of M, then M is a Fuz-⊕-S-mod.

Proof. We have M is A-Fuz-S-mod. Any Fuz-subs A, B of M ∋M=A+B and Fuz--supplement of A in B. (or, every F-sub of M has a A-Fuz-supplement in M. (i.e. for every v∈F(M) with μ+v=χM, there is a supplement σ of μ such that σ ⊆v). Also, every Fuz-sub of M is a Fuz-Li-above a direct summand of M. By Definition. 3.9, M is a Fuz-D_1-mod and from Lemma 3.6, M is a Fuz-L--mod. By Definition 3.1, "there is a Fuz-sub of M is a Fuz-supplement and it is a Fuz-direct summand". Thus M is Fuz-⊕-S-mod. From Lemma 3.6, "M is a Fuz-L-mod". By Definition 2.7, "there is a Fuz-sub of M is a Fuz-supplement and it is a Fuz-direct summand". Thus M is a Fuz-⊕-S- mod.

A module M is calle Fuzzy maximal submodule if and only if the quotient Fuzzy module A/M is a simple Fuzzy module.

Remark 3.11. If M is a Fuz-⊕-S-Mod, then the fuzzy of M divided radical of M is also Fuz-⊕-S-mod. On the other hand, we have fuzzy radical of M is the intersection of all Fuz-max-subs of M.

Note that the fact in Remark. 3.11 assistant to introduce the following theorem . But before that we need to define Fuz-D_1-module. Let N be a Fuzzy submodule of Fuzzy R-module M. If M has a decomposition M = M_1⊕M_2 such that M_1 ≤ N and M_2 ∩N is a Fuzzy small in M_2, then M is called Fuzzy-D_1-module.

Theorem 3.12. Let N be a Fuz-sub of an Fuz-mod M. If M=M⊕M_2 such that M_1 is a Fuz-sub of N and M_2∩N is a Fuz-small in M_2, then the quotient of M with respect to Fuz-Rad(M) is a Fuz-⊕-S-mod.

Proof. We have M is a Fuz-D_1-module. Therefore every Fuz-sub N of M is a ⊕ −Fuz-Li-above of M, so M is a Fuz-L-mod. So, is a Fuz-⊕-S-mod. Thus, M/ Fuz-Rad(M) is a Fuz-⊕-S-mod.

Theorem 3.13. Any ⊕-Fuz-sub-Li-above U of Fuz-Lin-compact-mod M is a Fuz-⊕-S-mod.

Proof. By Lemma 3.3, "U has A-Fuz-supplement in M. If H is a Fuz-sub of M and G is a Fuz-submodule of U such that M=G+H, then there is a Fuz-supplement H of U and M is A-Fuz-S- module". We have U is ⊕-Fuz-sub-Li-above of M. Thus M is Fuz-⊕-S-mod.

Example 3.14. Fuz-cyclic-mod over commutative Fuz-semiperfect ring R is a Fuz-⊕-S-mod, because any finitely generated module is cyclic and every cyclic over semiperfect is -⊕-S-mod**.**

Example 3.15. Every Fuz-f.g. free R-module over Fuz-semiperfect ring R with Fuz-radical (J) is a Fuz-⊕-S-mod.

Corollary 3.16. Every Fuz-supplement sub of R-module M is a Fuz-⊕-S-mod.

Proof. First, we need to prove that the F-radical of M equal zero. Suppose that Fuz- Rad(M) is not equal zero. There exists nonzero fuzzy element r belong to Fuz-Rad(M). We have Rr is a Fuz-supplement in M. Then M=Rr+H and Rr∩H is a Fuz-small in Rr ∋ H Fuz-sub of M. Since r belong to Fuz-radical of M, then Rr is a Fuz-small in M and H = M. Hence Rr is a Fuz-small in Rr which is impossible. Thus Fuz-radical of M equal zero. We have N is a Fuz-supplement, then K is Fuz-sub of M such that M=N+K and Fuz-(N∩K) = min (N∩K) is a Fuz-small in K. So Fuz-(N∩K) subset of 0 and hence Fuz-(N∩K) = 0. Hence M=N⊕K and so M is a Fuz-Sem- mod. Thus M is a Fuz-⊕-S-mod.

Corollary 3.17. Any direct sum of Fuz-sub over Fuz-V-ring R is a Fuz-⊕-S-mod.

Proof. Let M be a direct sum of Fuz-subs. Then M is Fuz-S-module. So, R is a left Fuz-Sem- ring. We have R is a Fuz-V-ring. Therefore from Lemma 3.8, M is a Fuz-⊕-S-mod.

Corollary 3.17. Every indecomposable Fuz-⊕-S-mod M with maximal Fuz-subs is Fuz-local.

Proof. Let N be a maximal Fuz-sub of M. Since M is indecomposable and Fuz-⊕-S-mod, N is fuzzy superuous

(Fuz-small). Hence N is the unique maximal Fuz-sub of M.

Corollary 3.19. Let M be a Fuz-⊕-S-mod. If M contains a maximal Fuz-sub, then M contains a local Fuz-direct summand. So is Fuz-hollow mod.

Proof. Let L be a max-Fuz-sub of M. Since M is Fuz-⊕-S-mod, there exists a Fuz-direct summand K of M such that K is a Fuz-supplement of L in M. Then for any proper Fuz-sub X of K, X is contained in L since L is a max-Fuz-sub and L+X is a proper Fuz-sub of M by minimality of K. Hence $X \leq L \cap K$ and X is Fuz-small in K. Thus K is a Fuz-hollow mod.

4. Conclusion

In this paper, the notion of Fuz-⊕-S-mod is defined and discussed. It has been proved that every fuzzy supplement submodule of R-module M is a Fuz-⊕-S-mod but the converse is not true in general for example Fuzzy Artinian and Fuzzy semisimple modules are Fuz-supplemented and then are Fuz-⊕-S-mod, but Z-module is Fuz-⊕-S-mod in Z, then every non-zero submodule has no supplements and so is not Fuzzy supplemented. Also, we proved that if N is a Fuz-Li- above a direct summand submodule of a fuzzy R-module M and has is a fuzzy linearly compact, then M is a Fuz-⊕-S-mod. We have observed that any direct sum of Fuz-subs over Fuz-V-ring R is a Fuz-⊕-S-mod of Z-module are also introduced and discussed.

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حول ضبابية الجمع المباشر للمودول المستكمل

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الخلاصة:

تم في هذا البحث اجراء عملية تعميم لمودول الجمع المباشر المستكمل. قمنا بدراسة ضبابية مودول الجمع المباشر المستكمل. كذلك تم دراسة بعض خواص هذا التعميم وعلاقته مع مودولات اخرى مثل المودول الضبابي شبه البسيط, مودول الرفع الضبابي, المودول الجزئي الضبابي و مودول الجمع المباشر المستكمل الضبابي.