TOPOLOGICAL ENTROPY AND THE PREIMAGE STRUCTURE OF MAPS

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ABSTRACT

Aim this article to provide an accessible introduction to the notion of topological entropy and (for context) its measure theoretic analogue, and then to present some work applying related ideas to the structure of iterated preimages for a continuous (in general non-invertible) map of a compact metric space to itself. These ideas will be illustrated by tow classes of examples, form circle maps and symbolic dynamics. My focus is on motivating and explaining definitions. Most results are stated with at most a sketch of the proof. The informed reader will recognize imagery from Bowen's exposition of topological entropy which I have freely adopted for motivation.

Measure-theoretic entropy

How mach can we learn from observations using an instrument with finite resolution?

A simple model of a single observation on a "state space" X is a finite partition $P=\{A_1,A_2,\ldots,A_N\}$ of X into atoms, grouping the point (states) in X according to the reading they induce on our instrument. A measure μ on X with total measure μ(X) =1 defines the probability of reading as :

$P_i= \mu(A_i)$, i=1,2,…,N

The entropy of the partition as the following

$$H(P) = -\sum_{i=0}^{N} Pi \log pi$$

Measures the a priori uncertainty about the outcome of an observation or conversely the information we obtain from performing the observation. The extreme values entropy among partition with a fixed number N of atoms are H(P) =0, when the outcome is completely determined (some P_i = 1, all others = 0), and H(P) = log N, when all outcomes are equally likely (Pi = $\frac{1}{N}$, i = 1,2,…,N).

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To model a sequence of observations at different times, we imaging a dynamical system generated by the (μ-measurable) map f: X→X , so the state initially at x∈ X evolves , after κ time intervals, to the state located at fκ (x), where

fκ =f,f,…f {k time}

An 0bservation made after κ time intervals is modeled by the partition f-κ[P] = {f-κ[A1] ,…, $f^{-\kappa}[A_N]$} ,where the κ^{th} iterated preimage of A⊂X is

f-κ[A1] ={x∈X / fκ(x) ∈A } .

Assuming that μ is an f-invariant measure (μ(f-1[A1] = μ (A)), the outcomes of observation made at different times are identically distributed. The joint distribution of n successive observations performed one time apart is modeled by the mutual refinement:

Pn = P∨ f-1 [P]∨ …f(n-1) [P]

Whose typical atom, Ai0 ∩ f-1[Ai1] … f-(n-1)[Ain-1], consists of the points with a given itinerary of length n with respect to P(i.e, fj(x) ∈ Aij, j = 0,..,n-1). The asymptotic average information per observation for sequence of successive observations

$$H(f,P) = \lim_{n\to\infty} \frac{1}{n} H(P_n)$$

Is the entropy of , f relative to P.

For example , suppose f:X→X is the restriction to the unit circle S1 = { x ∈C ⁄ | x |=1} of x→ x2. If we parameterize s1 by θ ∈ R using exp(θ)=e2πiθ ∈S1 , our map corresponds to θ→2θ (mod Z), the angle-doubling map. (Lebesgue) arc length measure is invariant under this map, and if P is a partition into tow semicircles say A1 = $\{0 \le \theta \le \frac{1}{2}\}$,A2= $\{\frac{1}{2} \le \theta \le 1\}$, then Pn is a partition into 2n intervals of equal arc length. Thus H(Pn) = n log2, so

H(f,P) = log2 .

Note that case the observations at different time are (probabilistically) independent knowing the itinerary of length dose not help us predict the next position of a random point .

An equivalent model of this situation comes from expressing the angle in binary notation :

$$\theta = \sum_{i=0}^{\infty} \frac{x_i}{2^{i+1}}, x_i \in \{0,1\}, i = 0,1.....$$

Which is ambiguous only on the Lebesgue-null set of dyadic rational values for θ. Up to this ambiguity, we have a bijection with the set {0,1}N of sequences x=x0,x1,… in {0,1}. For any finite sequence ω=ω0,…,ωn-1 ∈{0,1}n, the cylinder set

C(ω)={x∈{0,1}N ⁄ xi=ωi for i=0,1,…,n-1}

Of sequences which begin with ω corresponds to an are in S1 of length 2-n, and we can define a measure µ on {0,1}N via

µ(C(ω))=2-n for all ω of length n,

which is equivalent to arclength measure on S1. The angle-doubling corresponds to the shift map on sequences

s(x0x1x2…) = x1x2…

More generally, if ℵ is a finite set ("alphabet") and we assign a"weigh" P(a) ≥ 0 to each "letter" $a \in \aleph$ so that $\sum_{a\in\aleph} P(a) = 1$, then the formula

µ(C(ω0…ωn-1))=P(ω0)P(ω1)…P(ωn-1)

defines a probability measure on the space of sequences

ℵN ={x=x0x1…| xi ∈ℵ, i=0,1,…}

And the natural shift map on ℵN with this measure is called a Bernoulli Shift.

The partition P = {C(a)| a∈ℵ} has entropy

$$H(P) = -\sum_{a\in\aleph} P(a)\log P(a)$$

The refinement Pn consists of all cylinder sets C(ω) as ω ranges "words" ω=ω0…ωn-1 ∈ℵN of length | ω| = n, and straightforward calculation shows that successive observation are independent with

H(Pn) = nH(P), H(s,P) = H(P).

The quantity H(f,P) depends on our observational device. We obtain a device-independent measurement of the predictability of the measure theoretic model f:(X µ) →(X µ) by maximizing over all finite partitions this is the entropy of f with respect to µ;

$h_\mu(f)$ = sup{H(f,P) | P a finite measurable partition of X }.

It can be shown that the partition P of S1 into semicircles maximizes H(f,P) for the angle-doubling map so $h_\mu(f)$ =log2 in this case. For the general Bernoyll shift (determined by the weights P(a), a∈ℵ) ,the partition P = {C(a) | a∈ℵ) into cylinder sets again maximizes entropy, so in case

$$h_\mu(f) = -\sum_{a\in\aleph} P(a)\log P(a)$$

For example, the Bernoull shift corresponding to a biased coin flip, say $P(0)=\frac{1}{3}, P(0)=\frac{2}{3}$, has entropy $h_{\mu}(f)=\log 3-\frac{2}{3}\log 2 \angle \log 2.$

The idea of using shanna entropy in this way was suggested by Kolmogorov [kol58] (and refined by Sinnai [sin59], who showed that $h_\mu(f)$ invariant under measure-theoretic equivalence of dynamical systems and used this to prove the existence of non-equivalent Bernoulli shifts. Subsequences Ornstein[Orn74] showed that for a large class of ergodic systems (including Bernoulli shift[Orn70]) $h_\mu(f)$ is complete invariant , tow systems from this class are equivalent precisely if they have the same (measure-theoretic) entropy.

Topological entropy

Adler ,Konheim and Mc.Andlrew [AKM65] formulated an analogue of $h_\mu(f)$ when the measure space (X,μ) is replaced by a topological space and f is assumed continuous. They replaced the partition P with an open cover and the entropy H(P) with logarithm of the minimum cardinality of sub cover .They resulting topological entropy, h_{top} is an invariant of topological conjugacy between continuous maps on compact spaces.

A more intuitive formulation of htop(f) , given independently by Bowen [Bow7] and Dinaburg [Din70], uses separated sets in a (compact) metric space.

Separated sets

Let us again model observation vai instruments with finite resolution, but this time using a (compact) metric d on on our space X. We assume that our instrument can distinguish points x, $x' \in X$ precisely if $d(x,x') \geq \varepsilon$ for soe positive constant ε. A subset

$E \subset X$ is ε-separated if our instrument can distinguish the points of E. compactness puts a finite upper bound on the cardinality of any ε-separated set in X, and we can define

maxsep[d,ε,X] = max{card[E] | $E \subset X$ is ε-separated with respect to d }

On the circle, using d normalized arclenght

$$d(\exp(\theta),\exp(\theta'))=\min_{j\in Z}|\theta-\theta'+j\ |.$$

Any set of N equally spaced points

$$E_N(\exp(\theta))=\{\exp(\theta+\frac{j}{N})|\, j=0,\ldots N-1\}$$

Is a maximal ε-separated set whenever $1/N+1< \varepsilon \leq 1/N$, so

$\text{maxsep}[d,1/N,S^1] = \text{card}[E_N(x)]=N$

The sequence space $\aleph^N$ has a natural topology as the countable product of copies of the alphabet ℵ (which is given the discrete topology) this is captured in the mtric

$d(x,x') = 2^{-\delta(x,x')}$

where $\delta(x,x') = 1+\min\{i|\ x_i\neq x'_i\ \}$

Note that if two sequences x,x' have different initial words ω,ω' of length n (i.e ,$x\in C(\omega)$, $x'\in C(\omega)$, $|\omega|=|\omega'| = n$ and), $\omega\neq\omega'$), then $\delta(x,x') \leq n$,so C(ω) and C(ω') are at mutual distance at least 2^{-n}, and each such cylinder has diameter $2^{-(n+1)}$.It follows that a set consisting of one representative from each cylinder set $C(\omega),\omega \in \aleph^n$, is a maximal 2^{-n} _separated set, and since there are $(\text{card}|\aleph|)^n$ words of length n,

$\text{maxsed}[d,2\text{-}n, \aleph^N] = (\text{card}|\aleph|)^n$.

Bowen-Dinaburg deinution of topological entropy

Now we introduce dynamics map via a continuous map f: X→X, and ask about the resolution of n successive observations separated by unit time intervals. This is captured in the Bowen-Dinaburg metrics, defined for n = 1,2,… by

$$d_n^f(x,x')=\max_{o\leq j\leq n} d(f^i(x),f^i(x')).$$

Tow points $x,x' \in X$ cannot be distinguished by our sequence of measurements if they (n,ε)-shadow each other (i.e, $d(f_i(x),f_i(x')) < \varepsilon$ for i=0,…,1). So the points of $E \subset X$ are distinguished precisely if any tow $x \neq x' \in E$ have $d^f_n(x,x') \geq \varepsilon$ that is E is ε-separated with respect to d^f_n, or (n,ε)-separated.

The number of distinguishable orbit segments of length n is thus

$$\max esp[d_n^f, \varepsilon, X] = \max\{card[E] | E \subset X is (n,\varepsilon) - separated\}$$

For the angle-doubling map, note that if $d(x,x') \leq 1/4$ then $d(f(x),f(x')) = 2d(x,x')$. In particular if

$d(x,x') = 2^{-k}$

for some $k \geq 1$ then

$$d(f^j(x), f^j(x') = \begin{cases} 2^{j-k} & for \ \ j < k \\ 0 & for \ j \geq k \end{cases}$$

And, noting that $f(E_2^k(x)) = E_2^{k-1}(f(x))$, we see that $E_2^k(x)$ is

. 2^{-k}-separated with respect to d, and

. (n, ε)-separated for any $\varepsilon \leq 1/2$ if $n \geq k$

In particular, for $0 < \varepsilon < 1/2$ and $n > \log_{1/2} \varepsilon$,

$\max sep[d^f_n, \varepsilon, S_1] = card[E_2^n(x)] = 2^n$.

An (n, ε)- separated set is analogous to a collection of different itineraries of length n (with respect to some partition whose atoms have diameter ε). Since the number of conceivable itineraries grows exponentially with n, it is natural to $\{e_n\}$ of positive real numbers, we write

$$GR\{e_n\} = \lim_n \sup \frac{1}{n} \log e_n \ .$$

The complexity of the dynamics emanating from any subset $K \subset X$ is reflected in

$$h_{rop}(f,K) = \lim_{\varepsilon \to 0} GR\{\max sep[d_n^f, \varepsilon, K\}.$$

Our primary interest is is when K=X the topological entropy of f: X→X is

$$h_{top}(f) = h_{top}(f,x).$$

We have seen that the angle doubling map has topological entropy log2 , in fact the analogous angle-stretching maps $\zeta k : x \to xk$ ($k \geq 2$) satisfy

$h_{top}(\zeta_k) = \log_k$.

A beautiful general relation between measure-theoretic and topological entropy was established through the work of Goodwyn [Goo69], Dinaburg [Din70] and Goodman[Goo71].

Theorem 1 (Variational Principle for Entropy)

For f :X→X any continuous map on compact metric space,

$h_{top}(f) = \sup\{h_\mu(f) | \mu$ is an f –invariant Borl probability measure on X }.

One-sided subshifts

The shift map on the sequence space $\aleph^N$

$S(x_0x_1x_2\ldots) = x_1x_2\ldots$

Is a card[ℵ]-to one map, continuous with respect to the product topology. A subshif we mean the restriction f: X→X of the shift to the shift to a closed invariant subset $X \subset \aleph^N$, Such a is determined by its admissible words: for n = 1,2,…, let

$$W_n(X) = \{\omega = \omega_0 ... \omega_{n-1} \in \aleph^n | \exists \ x \in X, i \in N \ with \ \ x_{i+j} = \omega$$

Note that a word which appears starting at position I in $x \in X$ appears as the initial sub word of $f^i(x)$, which also belongs to X if X is shift-invariant. Thus $W_n(X)$ equal the set of words $\omega \in \aleph^n$ with $X \cap C(\omega)$ nonempty, and it follows that a maximal 2^{-n}-separated set $E_n \subset X$ results from picking one representative from each such nonempty intersection. Thus for $2^{-(k+1)} < \varepsilon < 2^{-k}$, E_{n+k} is a maximal (n,ε)-separated set, and $\max sep[d^f_n, \varepsilon, X] = card\ [W_{n+k}(X)]$

Given us for any subshift f:X→X

$$h_{top}(f) = \lim_{k \to \infty} GR\{card[W_{n+k+1}(X)]\} = GR\{card[W_n(X)]\}$$

We spell out the results of this calculation for severalexample.

Full shift : When $W_n=\aleph^n$, so $X=\aleph^N$, we have

$h_{top}(f)=GR\{card[\aleph]^n\}=\log card[\aleph]$.

"Golden Mean" Shift : Define X as the of al the set sequences of 0's and 1's in which 1 is never followed immediately by it self, so $W_2(X)=\{00.01,10\}$. If we list all words of length n, then the words of length n+1 come from either followed an arbitrary word of length n that ends in 0 with a 1. If we set

$\omega_n= card[W_n(X)]$.

We see there are ωn words of length n+1 which end in 0, and hence ωn+1 which end in 1. this gives the recursive relation

$\omega_{n+1} =\omega_n +\omega_{n-1}$.

Showing that ω_n grows at the same rate as the Fibonacci number Fn (in fact $\omega_n =F_{n+3}$) . This rate is known [LM95,p.101] to be the logarithm of the golden mean, so

$$h_{top}(f)=GR\{\omega_n\}=GR\{F_n\}=\log\ (\frac{1+\sqrt{5}}{2}).$$

A generalization of this example arises form any finite alphabet $\aleph=\{a_1,\ldots,a_n\}$ and a list $W_a\subset\aleph^2$ of allowed pairs; X is then defined as the set of all sequences in $\aleph^N$ for which every subword of length 2 belongs to W_a. This information can be encoded in a square transition matrix A of size N= card[ℵ] whose (i,j) entry is 1 (resp.0) if the word a_i,a_j belongs (resp. dose not belong) to W_a. Note that the (i,j) entry of a power A_k of the A equals the number of admissible words of length k+1 which begin with a_i and end with aj, so $\omega_n=card[W_n(X)]$ equals the sum $\|A_{n-1}\|$ of the entries of A_{n-1}, and we have

$h_{top}(f)=GR\{\omega_n\}=GR\{\|A^{n-1}\|\}=\log$ (spectral radius of A).

In the special case of the "golden mean "shift, we have

$$A=\begin{bmatrix}1 & 1\\ 1 & 0\end{bmatrix}$$

Whose characteristic polynomial, t^2-t-1, has the golden mean an its larger root.

Even shift: Let X be the set of sequences of 0's and 1's in which tow successive appearances of 1 are separated by a block of consecutive 0's of even length (with may be the empty block, of length zero). This is most easily described by giving a list W_d of disallowed words, in this case:

$Wd=\{1(0)^{2n+1}1|\ n=0,1,\ldots\}$

And specifying that X consists of all sequences in which no word from Wd appears (any where).

In general, such a description essentially specifies a basis of open subset of the complement $\aleph^N/X$. When such a list is (or can be made) finite, a recoding allows us to previous case by the allowed (or disallowed) pairs. This is called a subshift of finite type (or topological Markov chain).

The "even " shift is clearly not of finite type, as no lest on words of bounded length can detect long forbidden words. However, it can be shown [LM95,p.103] that in this case $card[W_n] = F_{n+3}-1$ (where again fn is the nth Fibonacci number), so the even shift has

$$h_{top}(f)=GR\{F_{n+3}-1\}=GR\{F_n\}=\log\ (\frac{1+\sqrt{5}}{2}).$$

Dyck shift: This beautiful example, first suggested by Krieger [Kr172] and named after an early contributor to the study of free groups and formal languages, codifies the rules of matching parentheses. As it is readily accessible in the literature, I give a detailed account based on ideas I learned Doris and Ulf Fiebig.

The alphabet consists of N pairs of matching left and right delimiters

$\aleph=\{\ell_1,\gamma_1,\ldots,\ell_N,\gamma_N\}$.

For example, when N = 2, we can think of γ

$\ell_1=$"(",$\gamma_1=$")" ,$\ell_2=$"{ ",$\gamma_2=$"}".

Call a word $\omega = \omega_0, \ldots, \omega_{2k-1}$ of even length balanced if its entries can be paired subject to:

. a pair of entries consists of a left delimiter to the of a matching right delimiter if $\acute{\omega}_\alpha$ is paired with $\acute{\omega}_\beta$, where $0 \leq \alpha \leq \beta \leq 2k-1$. then $\acute{\omega}_\alpha = \ell_i$ for some index I and $\acute{\omega}_\beta = \gamma_i$ for the some index.

. distinct pairs are nested or disjoint : given $\alpha < \beta$ as above, every intermediate $\acute{\omega}_\gamma$ $(\alpha < \gamma < \beta)$ is paired with some other intermediate $\acute{\omega}_\delta$ $(\alpha < \delta < \beta)$.

Note that a pairing of this type is unique if it exists. We regard the empty word as balanced.

Now we specify the (infinite) list of disallowed words as

$Wd = \{\ell_i b_{\gamma j} \mid b \text{ is a balanced word } i \neq j\}$.

The subshift on the set of sequences $Ɗ_N \subset \aleph^N$ in which no element of W_d appears is the (one –sided) Dyck shif on N pairs. When $N = 1$, W_d is empty, so $Ɗ_1$ is the full shif on two symbols, we will tacitly assume that $N \geq 2$.

Proposition 1: The Dyck shift f: $Ɗ_N \rightarrow Ɗ_N$ on N pairs has $h_{top}(f)=\log(N+1)$.

Proof:

An admissible word has the general form

$\omega = b_0\gamma_{i1}\, b_1\gamma_{i2} \ldots b_{k-1}\gamma_{ik}\, b_k\ell_{j1}b_{k+1}\ldots\ell_{jm}b_{k+m}$

where each b_α, $\alpha=0,\ldots,k+m$, is a (possibly empty) balanced subword.

And the $k \geq 0$ right delimiters which are not matched in ω all occur to the left of he $m \geq 0$ unmatched left delimiters in ω. This lead to a natural decomposition of any admissible word as a concatenation of three (possibly empty) subwords

$\omega = ABC$

where $B = b_k$ is balanced, while $A = b_0 \ldots \gamma_{ik}$ (resp. $C = \ell_{j+1} \ldots b_{k+m}$) ends (resp. starts) with an unmatched right (resp. left) delimiter.

To calculate the topological entropy, note first that every admissible word ω is the initial subword of at least N+1 admissible words of length $|\omega|$: the N word $\omega\ell_i$, i=1,…,N are always admissible if m=0. thus

$card[W_{n+1}] \geq (N+1)card\,[W_n]$

for all n, and so

$card(f)=GR\{card[W_n]\} \geq \log(N+1)$

To handle the opposite inequality, we estimate the cardinality of the sets A_n, B_n, C_n of admissible words of length n whose decomposition has only one nonempty factor, of the type indicated by the letter.

We begin with balanced words: since $B_n = \varphi$ for n odd, assume n=2p. To estimate $card[B_n]$, we note that the number of possible configurations of p "ℓ" ' s and "γ" 's in a balanced word of length n is balanced above by $\binom{n}{p}$, and for each such configuration, once we have assigned an index to each ℓ (which we can do in N^P ways), the uniqueness of the pairing insures that the word has been determined. Thus

$$card[B_n] \leq \binom{n}{p} N^p < (N+1),$$

where the last inequality is a consequence of the binomial theorem. We now consider the set Cn of words beginning with an unmatched left delimiter, noting that the initial length k subwords of any $\omega \in C_n$ itself belong to Ck. Given $\omega \in C_n$, we immediately have $\omega\ell_i \in C_{n+1}$ for $i = 1,\ldots,N$ and $\omega\gamma_i \in C_{n+1}$ provided that ω has at least tow unmatched left delimiters, the last of which is ℓ_i. This given us

$card[C_{n+1}] \leq (N+1)card[C_n]$

and since $card[C_1]=N$,

$card[C_n] \leq (N+1)card[C_n]^n$.

A similar estimate can be obtained for card[An], either by repeating the the argument or by noting the bijection between A_n and C_n obtained by reversing letter order and interchanging ℓ with γ (keeping indices).

Finally, to estimate card[Wn] we consider, for each ordered triple (i,j,k) of nonnegative integers summing to n, the of words of the from ω = ABC with |A| = i, |B| = j, and |C| = k, Since an arbitrary factoring is possible, the number of such words is

$\text{card}[A_i].\ \text{card}[B_j].\ \text{card}[C_k] \leq (N+1)^{i+j+k} = (N+1)^n$.

But the number of possible triple (i,j,k) summing to n is less than $(n+1)^3$, so

$\text{card}[W_n] \leq (n+1)^3 (N+1)^n$.

The growth rate of the right-hand quantity is log(N+1), so

$h_{top}(f) = \log(N+1)$.

■

Square-Free Sequences: An even more complicated subshift is defined by forbidding any subword to immediately follow a copy of itself:

$$W_d = \{\omega^2 = \omega\omega / \omega \in \aleph^+ = \bigcup_{k=1}^{\infty} \aleph^k\}.$$

An elementary argument shows that ℵ must have at least three letters for this to. Give a nonempty subshift. For three (or more) letters, there exist square and it is known [Bri63] that $h_{top}(f)>0$. Although there are some known bounds for the entropy [Gr01, She81a, She81b, SS82], a precise value has been determined.

Pointwise primage entropy

There is a curios asymmetry in the definitions of entropy in [1-2] which look only at the future behavior of points. When f is invertible, it turns out that the inverse map f^{-1} has the same entropy: for $h_{top}(f)$ this follows from the observation that x and x' (n,ε)-shadow each other under a homeomorphism f precisely if their $f^{(n-1)}$-images (n,ε)-shadow each other under f^{-1}.

However, when f is not invertible the iterated preimage f-n[x] of a point are in general sets rather than points, so the formulations in [2] cannot be reversed in time. In 1991, Langevin and Walczak [LW91] built on ideas from their earlier work with Ghys (on the " entropy"of a foliation) to formulate an invariant based on the behavior of preimages. We direct the interested reader to their original paper or to [NP99]for more details on this invariant, whose definition is rather involved, it is related to and often equals, the branch preimage entropy which we present in [5].

Instead we begin with a more accessible pair of invariant definition by Hurley [Hur95] in 1995. looking at the growth rate of the size of iterated preimages of a point, measured via the Bowen-Dinaburg metrics. The two invariant differ in the stage at which one globalizes the pointwise measurement by maximizing over $x \in X$:

$$h_p(f) = \sup_{x\in X} \lim_{\varepsilon\to 0} GR\{\ \max sep[d_n^f, \varepsilon, f^{-n}[x]]\}$$
$$h_m(f) = \lim_{\varepsilon\to 0} GR\{\max_{x\in X}\ \max esp[d_n^f, \varepsilon, f^{-n}[x]]\}$$

We refer to hp and hm collectively as pointwise preimage entropys both are invariant of topological cojugacy [NP99] and we have the truvual inequalities

$h_p(f) \leq h_m(f) \leq h_{top}(f)$.

There are examples for which either of these inequalities is strict any homeomorphism with $h_{top}(f) > 0$ works for the second inequality (since $f^{-1}[x]$ is a single point, both pointwise preimage entropies are zero) and an example for the first is given in [FFN03].

However, the thrust of our discussion in this section and the next is that there are many cases when the three invariant agree. (We will also see this from a different perspective in [5.2.].

For angle-doubling map, we note that the n^{th} iterated preimage of a point consists of 2^n equally spaced points:

$f^{-1}[x] = E_2{}^n(x_n)$

where x_n is any nth preimage of x for example if x = exp(0) we can take $x=\exp(2^{-n}0)$. Since this set is (n,ε)-

separated if $\varepsilon \le 2^{-n}$ (or $n \ge \log_{1/2}\varepsilon$), we have independent of $x \in S^1$,

$\max esp[d^f_n,\varepsilon,f^{-n}[x]]=card[f^{-n}[x]]=2^n$

so $h_p(f)=h_m(f)=\log 2$.

A similar argument gives the common value logk for $h_p(\zeta_k)$ amd $h_m(\zeta_k)$ where ζ_k is the angle-stretching map $x \mapsto x^k$, k=3,4,… .

Pointwise preimage entropy for subshifts

If $x \in X \subset \aleph^N$ is a point in the shift-invariant set X, its n^{th} predecessor set (in X) consists of all the words $\omega \in \aleph^n$ of length n such that the concatenation ωx also belong to X:

$P_n(x) = p_n(x,X) = \{ \omega \in \aleph^n / \omega x \in X\}$.

Note that by definition $p_n(x,X) \subset W_n(X)$. Clearly the nth iterated perimage of x under the subshift f: X→X is the set of all concatenations ωx ,$\omega \in p_n(x,X)$, so from our earlier calculations, when $0 < \varepsilon \le 1/2$ and $x \in X$

$\max esp[d^f_n,\varepsilon,f^{-n}[x]]=card[p_n(x)]$.

This immediately gives

$$h_p(f) = \sup_{x\in X} GR\{card[p_n(x)]\}$$

$$h_m(f) = GR\{\max_{x\in X} card[p_n(x)]\}$$

Again we trace the application of this through our examples of subshift:

Full shift: Clearly $pn(x,\aleph^N) = \aleph^n$ for all $x \in \aleph^N$, so

$h_p(f) = h_m(f) = \log card[\aleph]$.

Subshifts of Finite Type: When X is defined by the transition matrix A, the predecessor set of any $x \in X$ is determined by its initial entry, x_0. If we pick x0 so that this column sum grows (with n) at least as all the other columns, then any $x \in X$ beginning with x0 has a maximal growth rate, and this equals the growth rate of $\|A^n\|$, so

$h_p(f) = h_m(f) = GR\{\|A^n\|\} = \log$(spectral radius of A).

Even shift: The predecessor set of a sequence in the even shift is determined by te of the location first 1 in the sequence: if $x = 0^\infty$ then $p_n(x) = W_n(X)$, while if $x_k = 1$ and $x_i = 0$ for all i< k, then $\omega \in W_n(X)$ belongs to $p_n(x)$ if either $\omega = 0^n$ or ends with 10^ℓ, where ℓ has the same parity as k. Thus $p_n(x)$ is in one –to-one correspondence with the set of admissible words of length n+2 (resp. n+1) ending with 01 (resp. 1) if k is odd (resp. if k is even or $x = 0^\infty$), and our either considerations show that all of these sets grow at the rate

$$h_p(f) = h_m(f) = \log\left(\frac{1+\sqrt{5}}{2}\right).$$

Dyck Shift: If x is a sequence formed by concatenating infinitely many balanced words, then

$P_n(x,D_N) = W_n(D_N)$

So

$h_p(f) = h_m(f) = GR\{card[W_n(D_N)]\} = \log(N+1)$.

Square-Free Sequences: The predecessor sets in this subshift vary wildly from point to point (cf[5.1] and the tools used in the other cases tell us nothing about pointwise preimage entropy in this case.

The alert reader will have noted that in all the cases expect the last, the pointwise preimage entropies $h_p(f)$ and $h_m(f)$ agree not only with each other but also with the topological entropy $h_{top}(f)$. This is not accident.

Theorem 2 ([FFN03]) For any one-sided subshift f: X→X, if

$GR\{W_nX\}=\log\lambda$

Then there exists a point $p \in X$ such that $card[P_n(p,X)] \ge \lambda^n$ for all n = 1,2,… .

The argument for this rests on a combinational lemma concerning the growth of branches in a tree saying roughly that if we pick a "root" vertex and have, for some N, than λ^N vertices at distance N from the root, then for some k (depending on λ, N, and the maximum valence of vertices in the tree) there exists a vertex v such that for I = 1,…,k the number of vertices at distance I from v, in direction away from the root, is not least λ^i.

Entropy points

The phenomenon described for one-sided subshifts in the preceding section that the preimages of some point determine the topological entropy never occurs for homeomorphisms with positive topological entropy (e.g. most tow sided subshifts), since any preimage of point is still a single point. However, it is possible to resolve this cognitive dissonance via a calculation of topological entropy in the spirit of pointwise preimage entropy –looking at preimages of local stable sets instead of points.

For ε > 0, the ε-stable set of $x \in X$ under the map f: X→X is

$S(x,\varepsilon,f) = \{y \in X \mid d(f^i(x),f^i(y)) < \varepsilon \text{ for all } i \geq 0\}$.

(This is just the intersection of ε-stable with respect to the various Bowen-Dinaburg metrics.) We can define a kind of "ε-local preimage entropy" by

$$h_s(f,x,\varepsilon) = \lim_{\delta\to 0} GR\{\max sep[d_n^f, \delta, f^{-n}[S(x,\varepsilon,f)]]\}.$$

Recall that a map f: X→X is forward-expansive if for some expansiveness constant c > 0, every ε-stable set for $0 < \varepsilon \leq c$ is a single point (i.e., $S(x, \varepsilon, f) = \{x\}$ whenever $\varepsilon \leq c$ and $x \in X$. Every one-sided shift, as each of the angle-stretching maps on s1, is forward-expansive. Clearly forward-expansive maps,

$$h_p(f) = \sup_{x\in X} h_s(f,x,\varepsilon)$$

Whenever $0 < \varepsilon \leq c$, More generally though we have

Theorem 3 ([FFN03]) If X is a compact metric space of finite covering dimension, then for every continuous map f: X→X and every ε > 0,

$$\sup_{x\in X} h_s(f,x,\varepsilon) = h_{top}(f).$$

It is possible, adapting an argument of Mane [Man79], to show [FFN03] that forward-expansive of f: X→X implies finite covering dimension for X (if it compact metric), immediately implying the equality $h_p(f) = h_m(f) = h_{top}(f)$ in this case. Theorem 2 shows that for one-sided shifts, the supremum in Theorem 3 is actually a maximum. This leads us to consider the set of entropy point of a continuous map f: X→X, defined as

$$\varepsilon(f) = \{x \in X \mid \lim_{\varepsilon\to 0} h_s(f,x,\varepsilon) = h_{top}(f)\}.$$

Point of ε(f) are those near which the local "backward" behavior reflects the

topological entropy of f.

How big is the set ε(f) of entropy points for a general map f: X→X? For one-sided subshifts, ε(f) is always nonempty, but there are examples where it. Is nowhere dense in X, and there are examples of other continuous maps with ε(f) = φ [FFN03]. A number of conditions, given in [FFN03], imply ε(f) ≠ φ ,the most general of these was defined by Misiurewiez (modifying a notion due to Bowen) a continuous map f: X→X is asymptotically h-expansive if

$$\lim_{\varepsilon\to 0} \sup_{x\in X} h_{top}(f, S(x,\varepsilon,f)) = 0$$

In effect, this says that ε-stable sets for small ε > 0 look almost like point from the perspective of topological entropy. We have

Theorem 4 ([FFN03]) Every asymptotically h-expansive map on compact metric space has

ε(f) ≠ φ

Forward-expansive maps are automatically asymptotically h-expansive, but the latter class is far larger in particular.

Theorem 5 ([Buz97]) Every C^∞ diffeomorphism of a compact manifold is asymptotically h-expansive .

Branch preimage entropy

In formulating the pointwise preimge entropies, one focuses on the preimage sets $f^{-n}[x]$ individual points. These sets have natural tree-like structure. With preimage points as "vertices" and an "edge" from $z \in f^{-n}[x]$ to $f(z) \in f^{-(n-1)}[x]$, and one can try to examine the structure of branches in this tree sequences $\{z_i\}$

with z_0 = x and f(z_i+1) = z_i for all i. The idea of the Langavin-Walezak invariant [LW91], which compare points x,x' $\in$X by means of their respective branch structures, was used by Hurley [Hur95] to formulate an invariant that fits our general context and in many natural cases equals that defined by Langevin and Walezak [LW91].

A complication fir both formulations is that if map is surjective. Some brances may terminate at points with no preimage, to avoid this largely technical distraction, we will assume tacitly that f: X→X is a surjection.

Recall that for any compact metric space (X,d), there is an associated Haus-dorff metric $\Im$d which makes the collection $\Im$(X) of nonempty closed subsets of X into a compact metric space for $K_0,K_1 \in \Im(X)$,

$\Im d(K_0,K_1)=\max_{i=0,1}\{\sup_{x\in K_i}[\inf_{x'\in K_{i-1}} d(x,x')]\}$. Given f: X→X a continuous surjection, we can apply the Hausdorff extension to the Bowen-Dinburing metrics d^f_n to define a sequence of branch metrics on X via

$d^b_n(x,x') = \Im d^f_n (f^{-n}[x],f^{-n}[x'])$

That is x ω X is branch close to x$\in$X if every ranch at x is shadowed by some branch at x', and vice-versa. Applying the usual mechanism to these metrics yields the branch preimge entropy

$h_b(f)=\lim_{\varepsilon\to 0} GR\{\max sep[d_n^f,\varepsilon,X]\}$.

Standard arguments apply to show that topologically conjugate maps have equal branch preimage entropy. When f is a homeomorphism, this equals the topological entropy, but in general $h_p(f)$ acts very differently from $h_{top}(f)$ a number of general equalities (some times strict) for $h_b(f)$ [NP99].

One can think of $h_b(f)$ as measuring the homogeneity of the preimage sets of two points x,x'$\in$S1 under the angle-doubling mp are rotations of each other, yielding $d^b_n(x,x')$ = d(x,x') and hence $h_b(f)$ = 0, this argument has a natural extension to any self-covering map f : X→X.

Branch priemage entropy for subshits

Suppose that f : X→X is the restriction of the shift map to some (shift-invariant) closed subset $X\subset\aleph^N$. We have already seen that priemage sets can be identified with predecessor sets

$f^{-n}[x] = \{\omega_x \mid \omega\in P_n(x,X)\}$.

Suppose now that x,x'$\in$X have different $(n+k)^{th}$ predecessor sets say $\omega = \omega_0,\ldots,\omega_{n+k-1}\in P_{n+k}(x)\setminus p_{n+k}(x')$. which means that z = ωx belongs to $f^{n-(n-k)}[x]$, but for any $z'\in f^{-(n+k)}[x']$ we have z'= ω'x', where ω = $\omega'_0,\ldots,\omega'_{n+k-1}$ and $\omega'_I \neq \omega'_j$ for some j<n+k. If we let i = min(j,n), then the initial k-words of $f^i(z)$ and $f^i(z')$ are distinct so

$d^f_n(z,z') \geq 2^{-k}$.

and this shows that whenever $P_{n+k}(x) \neq P_{n+k}(x')$ as sets,

$d^b_n(x,x') \geq 2^{-k}$.

But if $\omega\in P_{n+k}(x)\cap P_{n+k}(x')$ then z = ωx and z' = ω'x' satisfy $d^f_n(z,z') \geq 2^{-k}$, it follows that

maxesp[d^b_n, 2^{-k}, X] = NP_{n+k}[X].

where NP_m[X] denoted the number of distinct mth predecessor sets $P_m(x)$ (as x ranges over X). So e have , for any one –sided subshift f : X→X,

$h_b(f)=\lim_{k\to\infty} GR\{NP_{n+k}[X]\}=GR\{NP_n[X]\}$. Here are details of this calculation for our earlier examples.

Full shift: Since

$P_n(x,\aleph^N)=\aleph^n$ for all $x\in\aleph^N$, $NP_n[\aleph^N]=1$ for all n and $h_b(f)=0$.

Subshift of Finite Type : We saw earlier that Pn(x) is determined by x_0, so $NP_n[x] \leq card[\aleph]$ for all n, and $h_b(f)$ = 0.

Sofic subshifts: We saw that even shift precisely two distinct nth predecessor sets for each n, so $NP_n[x]$ = 2 for all n $h_b(f)$ = 0. In general, a subshift f : X→X is called sofic if $NP_n[x]$ has a finite upper bound as n→∞ , Benjamin Weiss [We173] showed that f : X→X is sofic precisely if there is subshift of finite type g: Y→Y and a continuous surjection $\rho : Y \rightarrow X$ such that $\rho \circ g = f \circ \rho$ (i.e , f is a factor of g). All sofic subshift clearly have $h_b(f) = 0$.

Dyck shift: Any balanced word can precede any sequence in D_N: more generally if $\omega = ABC \in W_n$ (as in [2.2.3]) then if C is empty, $\omega \in Pn(x, D_N)$ for all $x \in D_N$. If $C \neq \theta$, the unmatched left delimiters in C must match the first unmatched right delimiters (if any) in x. To be precise, suppose $\omega \in W_n$ has $m \geq 0$ unmatched left delimiters, $\ell_j, \ldots, \ell_{jm}$ (reading left to-right in ω) and $x \in D_N$ has $0 \leq p \leq \infty$ unmatched delimiters, let q = min (m,p) ≤ n and suppose the first q unmatched right delimiters in x are $r_{s0}, \ldots, r_{sq}$ (reading left-to-right in x). Then $\omega \in Pn(x)$ precisely if the indices match, moving in opposite in and directions in x and ω :

$S_i = {}_{jm-i}$ for $0 \leq i \leq q$.

This show that the predecessor set Pn(x) determined by the indices of the first n (or if x has fewer) unmatched right delimiters in x, $NP_n[D_N]$thus equals the number of sequences of length n or less of indices from {1,…,N}, or

$$NP_n[D_N] = \sum_{i=0}^{n} N^i \leq (n+1)N^n$$

which has growth rate

$h_b(f, D_N) = GR\{(n+1)N^n\} = \log N$.

(For comparison recall that $h_{top}(f, D_N) = \log(N+1)$.)

Square-Free Sequences: We show as in [NP99] that if ℵ is an alphabet on has infinite branch preimage entropy.

Pick three distinguished letters from ℵ and $\beta = b_0 b_1 b_2 \ldots$

A square-free sequence in just these three letters. The complement $\aleph^*$of these letters in ℵ still has at least three letters, so we have the nonempty subset $X^* \subset X$ of square-free sequences which have no letter in common with β.

We will produce, for every subset $E \subset W_n(X^*)$ of square-free words in $\aleph^*$, a sequence $x_E \in X$ whose predecessor set in X intersects $W_n(X^*)$ precisely in E:

$P_n(x_E, X) \cap (\aleph^*)^n = P_n(x_E, X) \cap W_n(X^*) = E$.

When $E = W_n(X^*)$, $x_E = \beta$ works since for $A \in Wn(X^*)$ the sequence Aβ is square-free. Otherwise, consider the complement

$F = W_n(X^*) \setminus E = \{A_0, A_1, \ldots, A_k\}$

And for i = 0,…,k let $B_i = b_0 \ldots b_i$

Be the initial subword of length i+1 in β.

We can exclude A_0 from $P_n(x_E)$ by making sure the initial subword of x_E is $b_0 A_0 b_0$, for example if k = 0 (so $E = W_n(X_*) \setminus \{A_0\}$ we can take

$x_E = b_0 A_0 b_0 b_1 b_2 \ldots = B_0 A_0 \beta$:

any word $A \neq A_0$ in $\aleph^*$ which is square-free belongs to the predecessor set. If $k \geq 1$, we exclude A_1 (in addition to A_0) by making sure that an initial word ω_1 of x_E is followed by $A_1\, \omega_1$, we shall take

ω1 = b0A0bob1b2… = B0A0B1 so

$x_E = b_0 A_0 b_0 b_1 A_1 b_0 A_0 b_0 b_1 b_2 \ldots = \omega_1 A_1 \omega_1 b_2 \ldots = B_0 A_0 B_1 A_1 B_0 A_0 \beta$.

For I = 1,…,k-1, define ω_{i+1} recursively by

$\omega_{i+1} = \omega_i A i \omega_i b_{i+1}$.

noting that for $A \in W_n(X^*)$, $A\omega_{i+1}$is square-free precisely if A is distinct from $A_0, \ldots, A_i$. (The observation that $\omega_{i+1}$is itself square-free requires a little thought .)Note also that ω_{i+1} ends B_{i+1}. Thus, the sequence

$x_E = \omega_k b_{k+1} b_{k+2} \ldots$

is square-free, and its nth predecessor set intersects $W_n(X^*)$ precisely in E, as required.

This shows that the number $NP_n[X]$ of distinct n^{th} predecessor sets for X is bounded below by the number of distinct subsets of $W_n(X^*)$, or $2\omega_n$ (where $\omega_n = card[W_n(X^*)]$). But we know that ω_n has positive exponential growth rate (since X^* has positive topological entropy). And hence

$$h_b(f) = GR\{NP_n[X]\} \geq GR\{2^{\omega_n}\} = \left(\limsup_{n\to\infty} \frac{\omega_n}{n}\right) \log 2 = \infty$$

Hurle's inequalities

The main result of Hurle's paper [Hur95] is a beautiful inequalities relating pointwise, branch and topological entropy:

Theorem 6 ([Hur95]) For any continuous map f : X→X on a compact metric space,

$h_m(f) \leq h_{top}(f) \leq h_m(f) + h_b(f)$.

In particular, for any map with branch preimage entropy zero, pointwise preimage entropy automartically agrees with topological entropy. We have seen that this occurs for subshifts Theorem 2 appears to provide the only proof that $h_m(f) = h_{top}(f)$. Several other classes of maps are know to have $h_b(f) = 0$ (and hence $h_m(f) = h_{top}(f)$):

. A forward-expansive map on a compact manifold is automatically a self-covering map [HR69] and so has branch entropy zero (as noted earlier in this section).

. Any rational map f(z) =p(z)/q(z) (p,q polynomials) on the Riemann sphere has zero branch preimage entropy [LP92].

. If X is homeomorphic to a finite group (including the interval and circle) then every continuous map f : X→X has branch preimage entropy zero [NP99].

Natural extensions

Given f : X→X a continuous map on compact space, define the space

$\hat{X} = \hat{X}_f = \{\hat{x} = \ldots x_{-1}x_0x_1\ldots \in X^Z \mid f(x_i) = x_{i+1} \text{ for all } i \in Z\}$

9with the induced product topology) and the projection π: $\hat{X}$→X via $\pi(\hat{x}) = x_0$.

The image of the projection is the eventual range of f

$$\pi[X] = \bigcap_{i=0}^{\infty} f^i[X]$$

Which is homeomorphic to the quotient space $\hat{X}/\pi$. The shift map $\hat{f} : \hat{X} \to \hat{X}$

$[f(\hat{x})]_i = x_{i+1}$, $i \in Z$

Is a homeomorphism called the natural extension (or inverse limit) of f : X→X.

In effect, $\hat{X}_f$ separates the various prehistory's of points note that for $\hat{x} \in \hat{X}$, $x_0 = \pi(\hat{x})$ determines all xi with $i \geq 0$.

The natural extension of the angle-doubling map can be identified with the "solenoid" of samale [Shu80,4.9], [KH95,17.1], while the natural extension of a one-sided subshift $X \subset \aleph^N$ is the two-sided subshift $\hat{X} \subset \aleph^z$ specified by the same list of disallowed words. In general, $h_{top}(\hat{f}) = h_{top}(f)$.

Of course, topologically conjugate maps have topologically conjugate natural extension, but the converse is not always true. The following example was shown to me by Bob Burton.

Consider the coding $\Phi : \aleph^2 \to \vartheta$ which assigns to each word $\omega \in \aleph^2$ of length 2 in the alphabet $\aleph = \{0,1\}$ a letter $\Phi(\omega) \in \vartheta$ in the alphabet $\vartheta = \{1,2,3\}$ via

$\Phi(01) = 1$

$\Phi(11) = 2$

$\Phi(00) = \Phi(10) = 3$.

Any such coding induces a continuous map $\hat{h} : \aleph^z \to \vartheta^z$ via $\hat{h}(\hat{x}) = \hat{y}$, where

$y_i = \Phi(x_{i-1}x_i)$.

The image $\hat{h}[\aleph^z]$ is the subshift $\hat{X} \subset \vartheta^z$ with the transition matrix

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

Furthermore, yi determines xi, so hˆ is a homeomorphism between $\aleph^z$ and $X^\wedge \subset \vartheta^z$ which conjugates the shift maps on these spaces.

However, the one-sided subshift f : X→X defined by the transition matrix A cannot be conjugated to the (full) shift on $\aleph^N$, because for y = $y_0y_1\ldots \in$ X, f-1[y] has cardinality numerically equal to $y_0 \in \{1,2,3\}$, while every x $\in \aleph^N$ has precisely two preimages.

The two one-sided subshifts are both of finite type, so automatically satisfy $h_b(f) = 0$. But more generally, the following is true:

Theorem 7 If f : X→X and g : Y→Y are both forwrad-exponsive with topologically conjugate natural extension fˆ : Xˆ→Xˆ and gˆ : Yˆ→Yˆ ,then $h_b(f) = h_b(g)$.

this theorem was first conjugative by Bob Burton, with whom I unsuccessfully sougth a proof several years ago. I know of two arguments for this fact, both unpublished. One proceeds by analyzing the structure of conjugative between natural extensions (which for forward-expensive maps come from a kind of generalized coding) and using it to estimate the growth rate of maxsep[$d^b{}_n$,ε,X] for ε < c. The other is based on "lifting" $h_b(f)$ to fˆ by a trick similar to our replacement of points with local stable sets in [4].Unlike the situation three the when f is forward-expansive. Both arguments are due to Doris and Ulf Fiebig, with some contribution on my part to the first one.

Pressure and Hausdorff dimension

In the contex of an abstract "theormodynamic formalism" for dynamical systems, Ruelle [Rue73,Rue78] modified the concept of topological entropy, replacing the number maxsep[dbn,ε,X] of n-oribt segments with a "weighted" count,the weights coming from a function Φ, to get the topological pressure of Φ with respect to f. To precise given f : X→X a continuous map and Φ :X→R a continuous real-valued function, the sum of Φ along the n-orbit segment starting at $x \in X$ is denoted

$$S_n\Phi(x) = \sum_{i=0}^{n-1}\Phi(f^i(x))$$

And for ε > 0 we consider

$$N(f,\Phi,\varepsilon,n) = \sup_E \sum_{x\in E} e^{8_{n\Phi}}$$

The supremum taken over all (n, ε)-separated sets in X. The topological pressure of Φ with respect to f is then $P_f(\Phi) = \lim_{\varepsilon\to 0} GR\{N(f,\Phi,\varepsilon,n)\}$.

It can be shown that $P_f(\Phi)$ is either always finite or always infinite for all $\Phi \in C(X)$, the space of continuous real-valued functions on X, and when finite P_f:C(X)→R is monotone, convex and continuous. It is also clear that the topological pressure of the constant zero function is the topological entropy: $P_f(0) = h_{top}(f)$.

There is fascinating connection between topological pressure and hausdrff dimension of certain invariant sets. This connection was first noted, in the context of Fuchsian groups, in Bowen's last paper [Bow79] (published posthumously) and in generally referred Bowen's formula. For any strictly negative $\Phi \in C(X)$, the function t→ $P_f(t,\Phi)$ has a unique zero t_Φ. Ruelle showed [Rue82] that if f is $C^{1+\alpha}$ and J is a conformal repelled (J is the closure of some recurrent f-orbit, and the derivative multiplies the length of all vectors at $x \in J$ by a factor $\alpha(x)$, where $\alpha(x) > 1$ for all $x \in J$) then the Hausdorff dimension HD(J) of J equals t_Φ, where $\Phi(x) = -\log \alpha(x)$.

Analogous results for saddle sets of surface diffeomorphism were obtained by Manning et al [Man81, MM83]. A saddle set for a diffeomorphism of a surface is an invariant set A such that at each $x \in A$ there exist two independent vectors $v_+, v_- \in T_x A$ with $\|Df^n(v\pm)\|$ going to zero at a (uniform) exponential rate as $n \to \pm\infty$. Every point $x \in A$ then has invariant curve $W^*(x)$ (its stable manifold) which goes through x tangent to v_+. The prototype of this is the smale "horseshoe" ([Sh86,KH95]),where $v_\pm$ are coordinate vectors. The stable dimension at $x \in A$ of a saddle set A is the hausdorff dimension of the intersection of A with the stable manifold of x:

$sd(A,x) = HD(A \cap W^*(x))$.

If we define $Ø^s \in C(X)$ by

$Ø^s(x) = \log \|Df(v_+)\|$.

Then, under a few technical assumption we again have [MM83] Bowen's formula

$sd(A,x) = t_\Phi$.

The same formula was obtained for the C^2 version of the Henon map by Verjovsky and W_u[VW96].

When the map is not invertible, the situation becomes more complicated.Mihalescu [Mih01] showed that in a complex two-dimensional setting, the stable dimension of a saddle set for a holomorphic endomorphism (with no critical point in the set) has t_Φ.

as an upper bound, but the inequality can be strict. By taking account of the minimum number of priemages of points in A, Mihailescu and Urbanski [MU01] obtained a better upper bound on sd(A).

In the same paper [MU01], Mihailescu and Urbanski also obtained a lower bound using a new "entropy " invariant h-(f) which we shall sketch below theyshowed that this invariant for the restriction of to A, is a lower bound for the stable dimension times the super mum of $[Ø^s]$ on A. Subsequently [MU02] they defined two new notions of pressure $P^-_f(\Phi)$ and P_f-(Φ) and used B0wen type formulas to obtain lower and upper bound s for stable dimension.

A notion complementary to that of an ε-separated set is an ε-spanning set $E \subset X$ ε-spans X if every point of X is within distance < ε of some point of E.A(set-theoretically) maximal ε-separated subset of X automatically ε-spans X, and a minimal ε-spans set is $\frac{\varepsilon}{3}$-separated, so in all of our definitions of "entropy" we could replace maxsep[d, ε,X] with the number

$$\text{minspan}[d, \varepsilon, X] = \min \{\text{card}[E] \mid E \subset X \ \varepsilon\text{-spans}\}.$$

For the Mihailescu-Urbanski invariants it is more natural to work with this number.

The difference between $h_{top}(f)$ and $h_b(f)$, when phrased in terms of spanning sets, can be clarified (at least when f surjective) by noting that each n-branch $z_0,z_1,\dots,z_{n-1}$ of f-1 has a well-define "root" $x = z_0$ and "tip" $z = z_{n-1} \in f^{-n}[x]$ the latter determines the branch via $f(z_i) = z_{i-1}$. A set $E \subset X$ ε-spans X in the branch metric d^b_n if the collection of branches rooted at point in E, or in terms of "tip", $E_{f,n} = \{f^{-n}[x] \mid x \in E\} \subset \Im$ (X) , ε-spans Xf,n in the Hausdorff Bowen-Dinabrug metric $\Im d^n_f$ which is to say for any $x \in X$ we can find $x' \in E$ such that every branch rooted at one of x,x' is (n, ε)-shadowed by at least one branch rooted at the other. However if we consider branches without regard to their roots, merely asking for a collection of branches which includes an (n, ε)-shadow of every branch, we are simply asking for a collection of tips which ε-spans X in the Bowen-Dinaburg metric d^f_n, and so the usual machinery in this case leads to $h_{top}(f)$.

The Mihailescu-Urbanski definitions mix these two notions. Let us say that a collection of n-branches weakly ε-spans n-branches in X if for any $x \in X$ we find at least one n-branch at x which is (n, ε)-shadowed by one from our collection. Looking at "tips", this amounts to saying we have a collection $E' \subset X$ of tips such thst the minimum Bowen-Dinaburg distance d^f_n of any preimage set $f^{-n}[x]$, $x \in X$ from our set E' is at most ε. Dente the minimum cardinality of a set E' which weakly ε-spans n-branches in X by ω[f,n,ε,X]. and let

$$h_{\omega}(f) = \lim_{\varepsilon \to 0} GR\{\omega[f,n,\varepsilon,X]\}.$$

Note that since any set which (n, ε) spans X also weakly ε-spans n-branches in X by

$\omega\,[f,n, ,\varepsilon,X] \le \text{minspan}[d^f_n, \varepsilon, X]$

so $h_\omega(f) \le h_{top}(f)$.

Going further we say that a collection $E \subset X$ (of "roots") very weakly ε-spans n-branches X if the collection of all branches rooted at points of E weakly ε-spans n- branches in X. The minimum cardinality of a set which very weakly ε-spans n-branches in X, which we will denote v[f, n, ε , X], is bounded above by ω[f, n, ε , X], since if E' is the set of "tips" for a weakly ε-spanning set of n-branches, then the corresponding set $E = f^n[E']$ of "roots" is a very weakly ε-spanning set cardinality less then or equal to card[E']. Thus the "entropy" defined using v[f, n, ε , X],

$$h_v(f) = \lim_{l\varepsilon \to 0} GR\{v[f,n,\varepsilon,X]\}$$

satisfies

$$h_v(f) \le h_\omega(f) \le h_{top}(f)$$

Further more any set which ε-spans X in the branch metric d^b_n also weakly ε-spans n-branches in X,so

$v[f , n , \varepsilon, X] \le \text{minspan}[d^b_n, \varepsilon, X]$

Which implies $h_v(f) \le h_b(f)$.

to define the corresponding notion of pressure we set for $f : X \to X$ and $\Phi \in C(X)$.

$$P_f^-(\Phi) = \lim_{\varepsilon \to 0} GR\{\inf_{E'} \sum_{x \in E'} e^{S_n \Phi(z)}\}$$

Where the infimum is taken over sets E' of "tips" for collections which weakly ε-spans n-branches in X, and

$$P_f -(\Phi) = \lim_{\varepsilon \to 0} GR\{\inf \sum_{x \in E} \min_{z \in f^{-n[x]}} e^{S_n \Phi(z)}$$

Where the infimum is taken over sets E (of "roots") which very weakly ε-spans n-branches in X.

It can be shown [MU02] that these are invariant in the sense that if f : X→X and g : Y→Y are maps conjugated by the homeomorphism h : X→Y ($h \circ f = g \circ h$), then for any $\Phi \in C(X)$,

$$P_f^-(\Phi) = P_g^-(\Phi \circ h^{-1})$$
$$P_{f-}(\Phi) = P_{g-}(\Phi \circ h^{-1})$$

Note that when Φ is the constant zero function then $e^{S_n \Phi(z)}$ =1 for all $z \in X$ and $n \in N$ so

$$P_f^-(0) = h_\omega(f)$$
$$P_{f-}(0) = h_v(f).$$

The invariance of pressure implies the invariance of these "entropy" in [MU01,MU02] h_v (resp. h_ω) is denoted h-(resp. h^-).

The bounds on stable dimension given by Mihailes-Urbanski can then be stated as follows:

Theorem 8 ([MU02]) Suppose f is a homeomorphic Axiom A map of P^2 and A is a basic saddle set for f with no critical points of f. Let

$Ø^s(x) = \log \| Df(v_+) \|$

Where v_+ is the "contracting" vector at $x \in A$. and denote by ts (resp. t^s_-) the (unique) zero of the function $t \to P^-_f(t, Ø^s)$ (resp. $t \to P_{f-}(t, Ø^s)$). Then for all $x \in A$. $t^s_- \le sd(A,x) \le t^s$.

Other directions

I would like to close with some brief speculative comments two other possible directions of study in the spirit of preimage entropy :

Variantional Principle : The relation between measure-theoretic and topological entropy given by Theorem 1 has an extansion to topological pressure [Rue73,Wa176,Mis76]:

Theorem 9 (Variantional Principle) For any continuous map f : X→X on a compact metric space and any $\Phi \in C(X)$. $P_f(\Phi) = \sup_\mu \{h_p(f) + \int \Phi \, d\mu\}.$

Where the supermum is taken over all f-invariant Borel probability measures μ.

It is natural to ask whether there is an anglogue of this preimage entropy one needs to find an appropriate version of pressure and measure theoretic entropy probably based on the branch structure of preimages Mihailesau and Urbanski have some ideas results in this direction.

Semigroup Actions: The dynamics of a single map f : X→X can be viewed as an action of the semigroup N on X. Andrzej Bis[Bis02] has formulated analogues of the various primage entropies in the context of an action of any finitely-generated semigroup of continuous maps on a compact metric space. One might speculate that a combination of these ideas with those of Mihailesau and Urbanski might yield more general results on the dimension of fractals defined by iterated function systems.

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الانتروبي التبولوجيا وتركيب التطبيقات

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الخلاصة

الهدف من هذا البحث هو إن نكتب مقدمه وملاحظات سهله وواضحة عن الانتروبي التبولوجيا والتي هي مشابهة لنظرية القياس فأنها مشابهة أيضا لبعض القضايا التطبيقية المتعلقة بفكرة التركيب المتكرر للتطبيقات المستمرة (بشكل عام التطبيقات غير الانعكاسيه) من الفضاء المتري المتراص إلى نفسه. هذه الفكرة سوف تكون موضحه بسحب الصفوف بدراسة الصفوف المفصلة، إشكال التطبيقات الدائرية والرموز الديناميكية.وأخذنا بنظر الاعتبار شرح التعاريف المتعلقة بالموضوع وفيه أيضا إن تكون النتائج بشكل مهم وكذلك البراهين.