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EXPECTATION IN LOCAL FIELD

HASSAN H. EBRAHIM **MUNA A. MAHMOOD**

Tikrit University – College of .Science of computers and mathematics.

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ABSTRACT

In this paper we study the $p-$ adic numbers as an example to the local field. Also we present Evans' new definition of expectation for random variables in this field and we study the properties of expectation according to this definition.

1.Introduction

The local field is any locally compact, non -discrete , totally disconnected and topological field. The best known example of local field is the field of $p-$ adic numbers which is the completion of the rational numbers $\mathbb{Q}$ with respect to the $p-$ adic metric[2].

The $p-$ adic numbers was introduced first by Kurt Hensel in 1897 and there were other researchers work on this concept like Ostrowski in 1917 and Kurschak[5],[6] .

In 2007 Steven Evans and Tye Lidman introduced a new definition of expectation in local field. Our aim in this paper is to study the $p-$ adic numbers and use it to explain the construction of the new definition of expectation then prove that the properties of expectation in the classical definition are hold in the new one.

In what follows we will give an elementary definitions with their examples related to the concept of $p-$ adic numbers.

Definition 1-1 [7]

Let $\mathbb{Q}$ be a set of the rational numbers. For every prime p the p-adic absolute value of $x \in \mathbb{Q}$ is denoted by $|x|_p$ and defined as : $|x|_p = 0$ when $x = 0$, and $|x|_p = p^{-\ell}$ when $x = p^{\ell}\left(\frac{a}{b}\right)$, where a, b are non-zero integers which are not divisible by p and ℓ is any integer number.

* Corresponding author at: Tikrit University – College of .Science of computers and mathematics, Iraq.E-mail address: scianb@yahoo.com

Theorem 1-2 [7]

The map $|\cdot|_p : \mathbb{Q} \to \mathbb{R}$ has these following properties:

$|x|_p = 0 \Leftrightarrow x = 0$

$|xy|_p = |x|_p \, |y|_p$

… (1)

$|x+y|_p \le \max\{|x|_p, |y|_p\}$

For every $x, y \in \mathbb{Q}$. The third property is called non-archamedean or ultrametric inequality.

We prove these properties as follows:

Let $x = p^{\ell_1}\left(\frac{a_1}{b_1}\right)$, $y = p^{\ell_2}\left(\frac{a_2}{b_2}\right)$ then:

1) $|x|_p = 0 \Leftrightarrow \left|p^{\ell_1}\left(\frac{a_1}{b_1}\right)\right|_p = 0 \Leftrightarrow p^{\ell_1}\left(\frac{a_1}{b_1}\right) = 0 \Leftrightarrow x = 0$

$$|xy|_p = \left|p^{\ell_1}\left(\frac{a_1}{b_1}\right) \cdot p^{\ell_2}\left(\frac{a_2}{b_2}\right)\right|_p = \left|p^{\ell_1+\ell_2}\left(\frac{a_1 \cdot a_2}{b_1 \cdot b_2}\right)\right|_p$$
$$= p^{-(\ell_1+\ell_2)} = p^{-\ell_1} \cdot p^{-\ell_2} = |x|_p \cdot |y|_p$$

2) $|x+y|_p = \left|p^{\ell_1}\left(\frac{a_1}{b_1}\right) + p^{\ell_2}\left(\frac{a_2}{b_2}\right)\right|_p$

when $\ell_1 < \ell_2 : \le \left|p^{\ell_2}\left(\frac{a_1}{b_1}\right) + p^{\ell_2}\left(\frac{a_2}{b_2}\right)\right|_p$

$= \left|p^{\ell_2}\left(\frac{a_1b_2+a_2b_1}{b_1b_2}\right)\right|_p$, such that a1b2 + a2b1 is not divisible by p

$= p^{-\ell_2} = |y|_p$

when $\ell_1 > \ell_2 : \le \left|p^{\ell_1}\left(\frac{a_1}{b_1}\right) + p^{\ell_1}\left(\frac{a_2}{b_2}\right)\right|_p$

$= \left|p^{\ell_1}\left(\frac{a_1b_2+a_2b_1}{b_1b_2}\right)\right|_p$, such that a1b2 + a2b1 is not divisible by p

$= p^{-\ell_1} = |x|_p$

Therefore $|x+y|_p \leq \max\{|x|_p, |y|_p\}$. ■

From this theorem we have the following corollary.

Corollary 1-3 : Suppose that p is prime number then :

$|x+y|_p \leq |x|_p \vee |y|_p$, for every x, y $\in \mathbb{Q}$

Remark 1-4 [1] : We can write any non-zero rational number x uniquely as $x = p^{\ell}\left(\frac{a}{b}\right)$.

Now we will give some examples about p-adic absolute value :

Example 1-5

1) Let p be a prime number then:

$$\left|\frac{1}{4}\right|_p = \begin{cases} 4 & \text{when } p = 2 \\ 1 & \text{when } p \neq 2 \end{cases}$$

Proof(1):

$$\left|\frac{1}{4}\right|_2 = \left|2^{-2}\left(\frac{1}{1}\right)\right|_2 = 2^{-(-)2} = 2^2 = 4$$

let $p = 3 \Rightarrow \left|\frac{1}{4}\right|_3 = \left|3^0\left(\frac{1}{4}\right)\right|_3 = 3^0 = 1$

2) $\left|\frac{14}{15}\right|_2 = \left|2^1\left(\frac{7}{15}\right)\right|_2 = 2^{-1} = \frac{1}{2}$

3) $\left|\frac{14}{15}\right|_3 = \left|3^{-1}\left(\frac{14}{5}\right)\right|_3 = 3^1 = 3$

4) $\left|\frac{14}{15}\right|_5 = \left|5^{-1}\left(\frac{14}{3}\right)\right|_3 = 5^1 = 5$

5) $\left|\frac{14}{15}\right|_7 = \left|7^1\left(\frac{2}{15}\right)\right|_7 = 7^{-1} = \frac{1}{7}$

6) $|9|_{11} = \left|11^0\left(\frac{9}{1}\right)\right|_{11} = 11^0 = 1$

7) $|-1|_p = 1$ for all p.

Proof (7):

Suppose

$|-1|_p \neq 1 \Rightarrow \left|p^{\ell}\left(\frac{a}{b}\right)\right|_p \neq 1 \Rightarrow p^{-\ell} \neq 1 \Rightarrow -\ell \neq 0$

$\Rightarrow \ell \neq 0,$

that is contradiction since -1 has a uniquely form as $p^0\left(\frac{-1}{1}\right)$, that is $\ell = 0$. .

Proposition 1-6 [6]

The map $|\cdot|_p : \mathbb{Q} \to \mathbb{R}$ defined by:

$$|x|_p = \begin{cases} 0 & \text{if } x = 0 \\ p^{-\ell} & \text{if } x = p^{\ell}\left(\frac{a}{b}\right) \end{cases}$$

where $a, b \in \mathbb{Z}/\{0\}, \ell \in \mathbb{Z}$ and a, b are not divisible by p.

Is defined a norm on $\mathbb{Q}$ with respect to the field of rational numbers, this norm is called the p-adic norm .

Proof :

The properties in theorem (1-2) satisfying the conditions of the norm as follows:

1) $|x|_p = 0 \Leftrightarrow x = 0$ (from property 1)

2) $|\alpha x|_p = |\alpha|_p\, |x|_p$, for every $\alpha, x \in \mathbb{Q}$ (form property 2)

3) $|x+y|_p \leq |x|_p \vee |y|_p$ (from property 3)

$\leq \max\{|x|_p, |y|_p\}$

$\leq |x|_p + |y|_p$, for every $x, y \in \mathbb{Q}$.

The following example shows the p-adic norm:

Example 1-7

We can find the 5-adic norm of the numbers 75 , $\frac{2}{375}$, 3 as follows :

1) $|75|_5 = \left|5^2\left(\frac{3}{1}\right)\right|_5 = 5^{-2} = \frac{1}{25}$

2) $\left|\frac{2}{375}\right|_5 = \left|5^{-3}\left(\frac{2}{3}\right)\right|_5 = 5^3 = 125$

3) $|3|_5 = |4|_5 = |7|_5 = \left|\frac{12}{7}\right|_5 = 5^0 = 1$.

Recall that every normed space (X , $\|.\|$) is metric space where d(x , y)= $\| x - y \|$, for every x , y $\in$ X .

Remark 1-8 [7]

Since $|\cdot|_p$ is p-adic norm on $\mathbb{Q}$ then it is a metric space, this metric is called a p-adic metric , and defined by : $d_p(x,y) = |x-y|_p$ for every x , y $\in \mathbb{Q}$.

The following example will explain how can we fined the p-adic metric.

Example 1-9

1) In the 7-adic metric: $d_7(2,51) < d_7(1,2)$

$$d_7(2,51) = |51-2|_7 = |49|_7 = \left|7^2\left(\frac{1}{1}\right)\right|_7 = 7^{-2} = \frac{1}{49}$$

$$d_7(1,2) = |2-1|_7 = |1|_7 = \left|7^0\left(\frac{1}{1}\right)\right|_7 = 7^0 = 1.$$

2) In the 5-adic metric: $d_5(29,54) < d_5(13,5)$

$$d_5(29,54) = |54-29|_5 = |25|_5 = \left|5^2\left(\frac{1}{1}\right)\right|_5 = 5^{-2} = \frac{1}{25}$$

$d_5(13,5)=|13-5|_5=|8|_5=\left|5^0\left(\frac{8}{1}\right)\right|_5=5^0=1.$

Definition 1-10 [2]

The field $\mathbb{Q}_p$ of p-adic numbers are the completion of the field of rational numbers $\mathbb{Q}$ with respect to the p-adic metric. In the same way that the real numbers are the completion of the rational numbers with respect to the standard metric.

Definition 1-11 [4]

The closed unit ball around (0), $\mathbb{Z}_p=\{x\in\mathbb{Q}_p:|x|_p\le 1\}$ is the closure in $\mathbb{Q}_p$ of the integers $\mathbb{Z}$, called the p-adic integers. As $\mathbb{Z}_p=\{x\in\mathbb{Q}_p:|x|_p\le p\}$, the set $\mathbb{Z}_p$ is also open. Any other ball around (0) is of the form $\{x\in\mathbb{Q}_p:|x|_p\le p^{-k}\}=p^k\mathbb{Z}_p$ for some integer k.

Proposition 1-12 [3]

The p-adic numbers $\mathbb{Q}_p$ are local field.

Proof

1) Each of the balls $\mathbb{Z}_p$ is compact (since every closed set is compact [3]), hence $\mathbb{Q}_p$ is locally compact (because every point has a neighbourhood whose closure is compact).

2) $\mathbb{Q}_p$ is a topology on $\mathbb{Q}$:

a) $\emptyset\in\mathbb{Q}_p$.

b) $\mathbb{Q}\in\mathbb{Q}_p$.

c) Let $Z_{p1},Z_{p2}\in\mathbb{Q}_p$ such that $Z_{p1}=\{x\in\mathbb{Q}_p:|x|<p_1\}$ and $Z_{p2}=\{x\in\mathbb{Q}_p:|x|<p_2\}$

$Z_{p1}\cap Z_{p2}=\{x\in\mathbb{Q}_p:|x|<p_1\wedge|x|<p_2\}$

$=\{x\in\mathbb{Q}_p:|x|<p_1\wedge p_2\}$

Let $p_3=\frac{|p_1-p_2|}{4}\in Z_{p1}\cap Z_{p2}$

$\Rightarrow Z_{p3}=\{x\in\mathbb{Q}_p:|x|<p_3\}\Rightarrow Z_{p3}\in Z_{p1}\cap Z_{p2}$

$\therefore Z_{p1}\cap Z_{p2}\in\mathbb{Q}_p$.

d) Let $Z_{p1},Z_{p2},\cdots\in\mathbb{Q}_p$

$\Rightarrow Z_{p1}\cup Z_{p2}\cup\cdots=\{x\in\mathbb{Q}_p:|x|<p_1\vee|x|<p_2\vee\cdots\}$

$=\{x\in\mathbb{Q}_p:|x|<p_1\vee p_2\vee\cdots\}$

Let $p_4=\inf\{p_1,p_2,\cdots\}$

$\Rightarrow Z_{p4}=\{x\in\mathbb{Q}_p:|x|<p_4\}$

$\Rightarrow Z_{p4}\subset Z_{p1}\cup Z_{p2}\cup\cdots$

$\Rightarrow \cup_{i=1}^{\infty}Z_{pi}\in\mathbb{Q}_p$.

$\therefore\mathbb{Q}_p$ is a topology on $\mathbb{Q}$.

Now since $\mathbb{Q}_p$ is a topology on $\mathbb{Q}$, and $\mathbb{Q}_p$ is a field then $\mathbb{Q}_p$ is a topological field.

3) In particular, such a ball is an additive subgroup of $\mathbb{Q}_p$ and the balls are cosets of these closed and open subgroups. then the topology $\mathbb{Q}_p$ has a base of closed and open sets, and hence $\mathbb{Q}_p$ is totally disconnected.

$\therefore\mathbb{Q}_p$ is a local field.

From now on, we let $\mathbb{K}$ be a fixed local field. The following are the properties we need, there is a real-valued mapping $x\to|x|$ on $\mathbb{K}$ called the non-archimedean valuation with the properties (1). The third one of these properties is the ultrametric inequality or the strong triangle inequality. The map $(x,y)\to|x-y|$ on $\mathbb{K}\times\mathbb{K}$ is a metric on $\mathbb{K}$ which gives the topology of $\mathbb{K}$. A consequence of the strong triangle inequality is that if $|x|\ne|y|$, then $|x+y|=|x|\vee|y|$.

This latter result implies that for every "triangle" $\{x,y,z\}\subset\mathbb{K}$ we have at least two of the lengths $|x-y|,|x-z|,|y-z|$ must be equal and therefore often called the isosceles triangle property.

The valuation takes the values $\{q^k:k\in\mathbb{Z}\}\cup\{0\}$, where $q=p^c$ for some prime p and positive integer c (so that for $\mathbb{K}=\mathbb{Q}_p$ we have $c=1$).

Write $\mathbb{D}$ for $\{x\in\mathbb{K}:|x|\le 1\}$ (so that $\mathbb{D}=\mathbb{Z}_p$ when $\mathbb{K}=\mathbb{Q}_p$). Fix $\rho\in\mathbb{K}$ so that $|\rho|=q^{-1}$, then: $\rho^k\mathbb{D}=\{x:|x|\le q^{-k}\}=\{x:|x|<q^{-(k-1)}\}$ for each $k\in\mathbb{Z}$ (so that for $\mathbb{K}=\mathbb{Q}_p$ we could take $\rho=p$). The set $\mathbb{D}$ is the unique maximal compact subring of $\mathbb{K}$ (the ring of integers of $\mathbb{K}$).

Every ball in $\mathbb{K}$ is of the form $x+\rho^k\mathbb{D}$ for some $x\in\mathbb{D}$ and $k\in\mathbb{Z}$.

If $B = x + \rho^k \mathbb{D}$ and $C = y + \rho^\ell \mathbb{D}$ are two such balls, then:

- $B \cap C = \emptyset$, if $|x - y| > q^{-k} \vee q^{-\ell}$
- $B \subseteq C$, if $|x - y| \vee q^{-k} > q^{-\ell}$
- $C \subseteq B$, if $|x - y| \vee q^{-\ell} \leq q^{-k}$

In a particular, if $q^{-k} = q^{-\ell}$, then either $B \cap C = \emptyset$ or $B = C$, depending on whether or not $|x - y| > q^{-k} = q^{-\ell}$ or $|x - y| \leq q^{-k} = q^{-\ell}$.

Fix a probability space $(\Omega, \mathcal{F}, P)$. By $\mathbb{K}$-valued random variable, we mean a measurable map from Ω equipped with $\mathcal{F}$ into $\mathbb{K}$ equipped with its Borel σ-field. Let L^∞ be the space of $\mathbb{K}$-valued random variable X that satisfy
$\|X\|_\infty =$ esssup $|X| < \infty$. It is clear that L^∞ is a vector space over $\mathbb{K}$.

If we identify two random variables as being equal when they are equal almost surely, then:

$\|X\|_\infty = 0 \Leftrightarrow X = 0$

$\|cX\|_\infty = |c| \|X\|_\infty, \qquad c \in \mathbb{K}, X \in L^\infty$

$\|X + Y\|_\infty \leq \|X\|_\infty \vee \|Y\|_\infty, \qquad X, Y \in L^\infty$

The map $(X, Y) \to \|X - Y\|_\infty$ defines a metric on L^∞ (or, more correctly, on equivalence classes under the relation of equality almost every where), and L^∞ is complete in this metric [1] .

2.Construction of expectation in local field

Definition 2-1 [1]

Given $X \in L^\infty$, set $\epsilon(X) = \inf\{\|X - c\|_\infty : c \in \mathbb{K}\}$.

The expectation of the $\mathbb{K}$-valued random variable X is the subset of $\mathbb{K}$ given by:

$E[X] = \{c \in \mathbb{K} : \|X - c\|_\infty \leq \epsilon(X)\}$.

Theorem 2-2 [1]

The expectation of a random variable $X \in L^\infty$ is non-empty. It is the smallest closed ball in $\mathbb{K}$ that contains supp X (the closed supp of X).

Proof:

By the strong triangle inequality $(|x + y|_p \leq |x|_p \vee |y|_p)$:

$\|X - c\|_\infty \leq \|X\|_\infty \vee |c|$, and for $|c| > \|X\|_\infty$ then $\|X - c\|_\infty \leq \|X\|_\infty \vee |c| \Rightarrow \|X - c\|_\infty \leq |c| \Rightarrow \|X - c\|_\infty = |c|$.

Therefore, the infimum of $c \mapsto \|X - c\|_\infty$ over all $c \in \mathbb{K}$ is the same as the infimum over $\{c \in \mathbb{K} : |c| \leq \|X\|_\infty\}$, and any point $c \in \mathbb{K}$ at which the infimum of is achieved must necessarily satisfy $|c| \leq \|X\|_\infty$, that is $\epsilon(X) = \inf\{\|X - c\|_\infty : |c| \leq \|X\|_\infty\}$ and $E[X] = \{c : |c| \leq \|X\|_\infty, \|X - c\|_\infty = \epsilon(X)\}$.

Again by the strong triangle inequality, the function $c \mapsto \|X - c\|_\infty$ is continuous.

Consequently, $E[X]$ is non empty as the set of points at which a continuous function on a compact set attains its infimum.

Since $E[X]$ is a ball of radius (=diameter) $\epsilon(X)$ [if $\acute{c} \in E[X]$ and $\acute{\acute{c}} \in \mathbb{K}$ is such that $|\acute{\acute{c}} - \acute{c}| \leq \epsilon(X)$, then

$\acute{c} \in E[X] \Rightarrow \{\acute{c} \in \mathbb{K} : \|X - \acute{c}\|_\infty \leq \epsilon(X)\} \Rightarrow \|X - \acute{c} + \acute{\acute{c}} - \acute{\acute{c}}\|_\infty \leq \epsilon(X) \Rightarrow \|X - \acute{\acute{c}} + (\acute{\acute{c}} - \acute{c})\|_\infty \leq \epsilon(X)$,

since $\|X - \acute{\acute{c}}\|_\infty \neq \|\acute{\acute{c}} - \acute{c}\|_\infty$, then by strong triangle inequality:

$\|X - \acute{\acute{c}} + (\acute{\acute{c}} - \acute{c})\|_\infty = \|X - \acute{\acute{c}}\|_\infty \vee |\acute{\acute{c}} - \acute{c}|_\infty \leq \epsilon(X) \Rightarrow \|X - \acute{\acute{c}}\|_\infty \leq \epsilon(X)$

$\Rightarrow \acute{\acute{c}} \in E[X]$.

Thus $E[X]$ is a (closed) ball in $\mathbb{K}$. (where we take a single point as being a ball)].

If $x \in$ supp X and x $\notin E[X]$, and c is any point in $E[X]$, then:

$\|x - c\|_\infty = \|x - X + X - c\|_\infty$

$= \|(-1)(X - x) + (X - c)\|_\infty$

Since $\|X - x\|_\infty \neq \|X - c\|_\infty$, then by the strong triangle inequality:

$= \|(-1)(X - x)\|_\infty \vee \|X - c\|_\infty$

$= |-1| \|X - x\|_\infty \vee \|X - c\|_\infty$

$= \|X - x\|_\infty \vee \|X - c\|_\infty$

Since $x \notin E[X] \Rightarrow \|X - x\|_\infty > \epsilon(X)$

$\Rightarrow \|X - x\|_\infty \vee \|X - c\|_\infty > \epsilon(X)$

$\Rightarrow \|x - c\|_\infty = |x - c| > \epsilon(X)$.

Since $x \in \operatorname{supp} X \Rightarrow \{\ \omega \in \mathbb{R}^n: X(\omega) \neq 0\ \}$, and $x = X(\omega) \Rightarrow \|X(\omega) - c\|_\infty > \epsilon(X) \Rightarrow \|X - c\|_\infty > \epsilon(X)$, contradicting the definition of $E[X]$, then $x \in \operatorname{supp} X$ and $x \in E[X]$. Thus $\operatorname{supp} X \subseteq E[X]$, hence if the smallest ball containing $\operatorname{supp} X$ is not $E[X]$, it must be a ball contained in $E[X]$ with diameter $r < \epsilon(X)$. However if c is any point contained in the smaller ball, then $|x - c| \leq r$ for all $x \in \operatorname{supp} X$. Contradicting the definition of $\epsilon(X)$. [because $\epsilon(X) = \inf\{\|X - c\|_\infty : c \in \mathbb{K}\}$ and $|x - c| \leq r < \epsilon(X)$]. ■

According to theorem 2-2 we give the following proposition :

Proposition 2-3

For each $X \in L^\infty$ then

$$E[X] = \{k \in \mathbb{K}: \|X(c) - k\|_\infty \leq \epsilon(X), \forall\ c \in \operatorname{supp} X\}.$$

Proof:

$\operatorname{supp} X = \{c \in \mathbb{R}^n: X(c) \neq 0,\ X$ is continuous functions$\}$ $E[X] = \{k \in \mathbb{K}: \|X - k\|_\infty \leq \epsilon(X)\}$, such that

$\epsilon(X) = \inf\{\|X - k\|_\infty : k \in \mathbb{K}\}$

$\Rightarrow E[X] = \{k \in \mathbb{K}: \|X(c) - k\|_\infty \leq \epsilon(X)\}$.

if $X(c) = 0$

then $E[0] = \{k \in \mathbb{K}: \|0 - k\|_\infty \leq \epsilon(0)\}$

$= \inf\{\|0 - k\|_\infty : k \in \mathbb{K}\}$

$= \inf\{\|k\|_\infty : k \in \mathbb{K}\}$

$\Rightarrow E[0] = \{k \in \mathbb{K}: \|k\|_\infty \leq \inf\{\|k\|_\infty\}\}$

That is contradiction, because $\|k\|_\infty \geq \inf\|k\|_\infty$

$\Rightarrow X(c) \neq 0 \Rightarrow c \in \operatorname{supp} X$

$\therefore\ E[X] = \{k \in \mathbb{K}: \|X(c) - k\|_\infty \leq \epsilon(X), \forall\ c \in \operatorname{supp} X\}$. ■

Lemma 2-4 [5]

If A is compact subset of the local field, then the smallest ball that contains A is $a + \{x: |x| < r\}$, where a is any point in A and $r = \max\{|x - a|: x \in A\} = \max\{|x - y|: x, y \in A\}$.

Lemma 2-5 [3]

If $\acute{A}$ and $\acute{\acute{A}}$ are compact subsets of the local field and $\acute{B}$ (respectively, $\acute{\acute{B}}$) is the smallest ball that contains $\acute{A}$ (respectively, $\acute{\acute{A}}$), then $\acute{B} + \acute{\acute{B}}$ is the smallest ball that contains $\acute{A} + \acute{\acute{A}}$.

Proof:

Choose $\acute{a} \in \acute{A}$ and $\acute{\acute{a}} \in \acute{\acute{A}}$. From lemma 2-2-4 above,

$\acute{B} = \acute{a} + \{x: |x| < \acute{r}\}$, and $\acute{\acute{B}} = \acute{\acute{a}} + \{x: |x| < \acute{\acute{r}}\}$

where

$\acute{r} = \max\{|x - \acute{a}|: x \in \acute{A}\}$, and

$\acute{\acute{r}} = \max\left\{|x - \acute{\acute{a}}|: x \in \acute{\acute{A}}\right\}$.

Similarly, the smallest ball containing $\acute{A} + \acute{\acute{A}}$ is $(\acute{a} + \acute{\acute{a}}) + \{x: |x| < r\}$,

where

$r = \max\left\{|(\acute{x} + \acute{\acute{x}}) - (\acute{a} + \acute{\acute{a}})|: \acute{x} \in \acute{A}, \acute{\acute{x}} \in \acute{\acute{A}}\right\}$.

$$\acute{B} + \acute{\acute{B}} = (\acute{a} + \acute{\acute{a}}) + \{x: |x| < \acute{r} + \acute{\acute{r}}\}$$
$$= (\acute{a} + \acute{\acute{a}}) + \{x: |x| < r\}.$$

$\therefore \acute{B} + \acute{\acute{B}}$ is smallest ball that contains $\acute{A} + \acute{\acute{A}}$. ■

Now we investigate the classical properties of expectation in Evans' definition :

Proposition 2-6

Let X and $Y \in L^\infty$ then:

1) $E[aX + b] = a\ E[X] + b$, such that a, b are constants.

2) $E[X + Y] \subseteq E[X] + E[Y]$, with equality when X and Y are independent.

3) $E[X \cdot Y] = E[X] \cdot E[Y]$, where X and Y are independent.

Proof:

1) We want to prove that: if $c \in E[aX + b] \Rightarrow c \in a\ E[X] + b$,that is: if $c_1 \in a\ E[X] + b \Rightarrow \exists\ c \in \mathbb{K}: \|X - c\|_\infty \leq \epsilon(x)$ such that $c_1 = ac + b$.

$E[X] = \{c \in \mathbb{K}: \|X - c\|_\infty \leq \epsilon(x)\}$

$= \{c \in \mathbb{K}: a.\|X - c\|_\infty \leq a.\epsilon(x)\}$

$= \{c \in \mathbb{K}: \|aX - ac\|_\infty \leq \epsilon(ax)\}$

$= \{c \in \mathbb{K}: \|aX + b - ac - b\|_\infty \leq \epsilon(ax)\}$

$= \{c \in \mathbb{K}: \|(aX + b) - (ac + b)\|_\infty \leq \epsilon(ax)\}$

Let $c_1 = ac + b \Rightarrow c_1 \in a\ E[X] + b$

$= \{c_1 \in \mathbb{K}: \|(aX + b) - c_1\|_\infty \leq \epsilon(ax)\}$
$\Rightarrow c_1 \in E[aX + b]$ and $c_1 \in a\, E[X] + b$
$\therefore E[aX + b] = a\, E[X] + b.$

2) Write supp X and supp Y for the supports of two random variables X and Y. Regardless of the dependence between X and Y, it is always the case that
$\mathrm{supp}(X + Y) \subseteq \mathrm{supp}\, X + \mathrm{supp}\, Y,$ and there is equality when X and Y are independent.

So, when X and Y are independent, we have
$\mathrm{supp}\,(X + Y) = \mathrm{supp}\, X + \mathrm{supp}\, Y.$

From theorem 2-2-2, $E[X]$ is the smallest ball that contains the support of X.
$\therefore E[X + Y]$ is the smallest ball that contains $\mathrm{supp}\,(X + Y) = \mathrm{supp}\, X + \mathrm{supp}\, Y.$

By the same theorem, $E[X]$ is the smallest ball that contains $\mathrm{supp}\, X,$ and $E[Y]$ is the smallest ball that contains supp Y.

From lemma 2-2-5 above we have,
$E[X] + E[Y]$ is the smallest ball that contains
$\mathrm{supp}\, X + \mathrm{supp}\, Y.$
$\therefore E[X + Y] = E[X] + E[Y].$

For example:

Suppose that X is any non-constant random variables (so that supp X does not consist of a single point), and put $Y = -X$ then $X + Y = 0$ So $\mathrm{supp}(X + Y) = \{\emptyset\}$ whereas
$\mathrm{supp}\, X + \mathrm{supp}\, Y = \mathrm{supp}\, X - \mathrm{supp}\, Y$
$= \{a - b : a \in \mathrm{supp}\, X \text{ and } b \in \mathrm{supp} X\}$ which will not consist of just a single point.
$\therefore\ \mathrm{supp}\,(X + Y) = \mathrm{supp}\, X + \mathrm{supp} Y.$

3) when X and Y are independent, we have
$\mathrm{supp}\,(X \cdot Y) \subseteq \mathrm{supp}\, X \cdot \mathrm{supp}\, Y$

From theorem 2-2-2, $E[X]$ is the smallest ball that contains the support of X.
$\Rightarrow \therefore E[X \cdot Y]$ is the smallest ball that contains
$\mathrm{supp}\,(X \cdot Y) \subseteq \mathrm{supp}\, X \cdot \mathrm{supp}\, Y$

By the same theorem, $E[X]$ is the smallest ball that contains supp X and $E[Y]$ is the smallest ball that contains supp Y.
$\therefore E[X] \cdot E[Y]$ is the smallest ball that contains
$\therefore E[X \cdot Y] = E[X] \cdot E[Y] . \mathrm{supp} X \cdot \mathrm{supp}\, Y.$

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التوقع في الحقل المحلي

حسن حسين ابراهيم منى عبد اللطيف محمود

الخلاصة:

درسنا في هذا البحث p-adic numbers كمثال على الحقل المحلي Local field. كما عرضنا تعريف ايفانس الجديد للتوقع Expectation للمتغيرات العشوائية في هذا الحقل وكذلك درسنا خواص التوقع حسب هذا التعريف .