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Pseudo-Extending Modules And Related Concepts

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ABSTRACT

In this work, we introduce the concept of pseudo-extending modules as a generalization of extending modules. Many characterization, and properties of pseudo-extending module are obtained. Characterizations of pseudo-extending module in some classes of modules are given. Modules imply pseudo-extending module are studied.

Introduction

Throughout this paper R will denote an associative ring with identity and all R-modules are unitary (left) R-module. An R-module M is called extending if every submodule of M [5], where we said that a nonzero submodule K of M is essential in M if $K \cap L \neq (0)$ for every non zero submodule L of M [4]. Extending modules have been studied recently by several author among them, Dung, N.V., Smith, P.F.,and Wisbauer [3], Mohamed, S.H. and Muller, B.J. [9].

In this work the concept of pseudo-extending modules is given as a generalization of extending modules. An R-module M is called a pseudo-extending, if every pseudo stable submodule is essential in a direct summand of M.

A non zero submodule N of An R-module M is called pseudo-stable, if for each R-monomorphism $f: N \to M, f(N) \subseteq N$ [1]. An R-module M is called a uniform, if everynon-zero submodule of M is essential in M [4].

§1:Pseudo-Extending Modules

In this section we introduce and study a class of modules which is a generalization of extending modules.

Definition 1.1

An R-module M is called a pseudo-extending, if every pseudo-stable submodule of M is essential in a direct summand of M.

A ring R is a pseudo-extending, if R is a pseudo-extending R-module.

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Remarks and examples 1.2

1) Every extending R-module is a psudo-xtending.
2) Every quasi-injective R-module is pseudo-extending.
3) Every uniform R-module is pseudo-extending
4) Z_n as a Z-module for any $n > 1$ is pseudo-extending
5) Every semi-simple R-module is pseudo-extending.

The following theorem gives a characterization of pseudo-extending modules.

Theorem 1.3

An R-module M is pseudo-extending if and only if for each pseudo-stable submodule A of M, there is a decomposition $M = M_1 \oplus M_2$ such that $A \subseteq M_1$ and $A \oplus M_2$ is essential submodule of M.

Proof

$(\Rightarrow)$ Let A be a pseudo- stable submodule of M, then A is essential in a direct summand of M say K . That is $M = K \oplus H$ for some submodule H of M. Since A is essential in K and H is essential in H, then $A + H$is essential in M.

$(\Leftarrow)$ Let A be a pseudo-stable submodule of M , then by hypothesis, there is a decomposition $M = M_1 \oplus M_2$ such that $A \subseteq M_1$ and $A + M_2$ is essential submodule of M. We claim that A is an essential submodule of M_1 , thus K_1 is a non-zero submodule of M_1, thus K_1 is a submodule of M. Since $A + M_2$ is essential in M, then $(A + M_2) \cap K_1 \neq (0)$. Let $l = a + m$ be a non-zero element of K_1, a in A and m in M, thus $m = l - a$ which implies that $m \in M_1 \cap M_2 = (0)$

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so we have $l = a \in K_1 \cap A \neq (0)$, and hence A is essential in M_1. Therefore M is pseudo-extending module.■

Recall that a submodule N of an R-module M lies under a direct summand of M, if there exists a direct decomposition $M = M_1 \oplus M_2$ with $N \subseteq M_1$ and N is essential in M.[10]

Corollary 1.4

Let M be an R-module. Then M is a pseudo extending if and only if every pseudo- stable submodule of M lies under a direct summand of M.

Proof:

$(\Rightarrow)$ Let N be a pseudo-stable submodule of M, and $M = M_1 \oplus (0)$ be a direct decomposition of M. Then by theorem 1.3 we have $N \subseteq M$ and $N \oplus (0)$ is essential in M . Hence N is lies under a direct summand of M.

$(\Leftarrow)$ Trivial.■

The following theorem is another characterization of pseudo-extending modules.

Theorem 1.5

Let M be an R-module. Then M is a pseudo-extending if and only if for each pseudo stable submodule N of M, there exists an idempotent $f \in End_R(E(M))$ such that N is essential in $f(E(M))$ and $f(M) \subseteq M$

Proof

$(\Rightarrow)$ Let N be a pseudo-stable submodule of M. then there is a direct summand D of M such that N is essential in D. Hence $M = D \oplus H$ for some submodule H of M. Thus we have $E(M) = E(D) \oplus E(H)$[12]. Let $f: E(M) \to E(D)$ be the projection homomorphism of $E(M)$ onto E(D). Hence f is an idempotent [13]. Thus, we have $f(M) \subseteq D \oplus H \subseteq D \subseteq M$. Now, since N is essential in D and D is essential in $E(D)$, then N is essential in $E(D) = f(E(M))$.

$(\Leftarrow)$ Let N be a pseudo-stable submodule of M, then by hypothesis there is an idempotent $f \in End_R(E(M))$ such that N is essential in $f(E(M))$ and $f(M) \subseteq M$. Now since M is essential in M, then, we have $N = N \cap M$ is essential in $M \cap f(E(M)) = f(M)$. But $f(M)$ is a direct summand of M [13] . Hence M is a pseudo-extending module.

In this section, we give many characterizations of pseudo-extending modules in some classes of modules.

Before that we give the first characterization, we must introduce the following lemma.

Lemma 2.1

If N is a pseudo-stable submodule of M, then $cl(N)$ is a pseudo-stable submodule of M.

Proof:

Let $f: cl(N) \to M$ be an R-monomorphism, and $m \in cl(N)$, then $Jm \subseteq N$ for some essential ideal J of R. Now, consider $Jf(m) = f(Jm) \subseteq f(N) \subseteq N$. Thus $f(m) \in cl(N)$. Hence $f(cl(N)) \subseteq cl(N)$. Therefore $cl(N)$ is a pseudo-stable submodule of M.■

Proposition 2.2

Let M be an R-module such that each submodule of M is essential in its closure, then M is a pseudo-extending if and only if for any submodule N of M with pseudo-stable closure is essential in a direct summand of M.

Proof:

$(\Rightarrow)$ Let K be a submodule of M , with pseudo-stable closure. Then $cl(K)$ is essential in direct summand of M say D. Then by hypothesis K is essential in $cl(K)$ and $cl(K)$is essential in D . Hence K is essential in D [4]

$(\Leftarrow)$ Let K be a pseudo-stable submodule of M, then by Lemma 2.1 $cl(K)$ is a pseudo-stable submodule of M, thus by hypothesis K is essential in $cl(K)$ and hence K is essential in a direct summand of M. Therefore M is pseudo-extending.■

Recall that an R-module M is a non-singular if $Z(M) = \{m \in M : Im = 0 \text{ for some essential ideal } I \text{ in } R\} = (0)$ [8].

Theorem2.3

Let M be a non-singular R-module. Then the following statements are equivalent.

1. M is pseudo-extending.

2. Every closed pseudo-stable submodule of M is a direct summand of M.
3. Every pseudo-stable submodule of M is essential in a pseudo-stable direct summand of M.

Proof:

$(1) \Rightarrow (2)$ let K be a closed pseudo-stable submodule of M, then K is essential in a direct summand D of M. But K is closed submodule in M, so K=D. Hence K a direct summand of M.

$(2) \Rightarrow (3)$ Let N be a pseudo-stable submodule of M. Since M is non-singular modules, thus there exists a closed submodule $cl(N)$ such that N is essential in $cl(N)$[8] Since N is a pseudo-stable submodule of M, then $cl(N)$ is a pseudo-stable submodule of M. Thus, by hypothesis $cl(N)$ is a direct summand of M. That is N is essential in a pseudo-stable direct summand $cl(N)$ of M.

$(3) \Rightarrow (1)$ Trivial. ■

Before we give the next characterization, we introduce the following definition.

Definition 2.4

A submodule N of an R-module M is called hyperpseudo-stable, if $E(N)$ is a pseudo-stable submodule in $E(M)$. M is called fully hyperpseudo-stable if each submodule of M is hyperpseudo-stable.

Theorem 2.5

Let M be a fully hyperpseudo-stable R-module. Then M is pseudo-extending if and only if $M \cap A$ is a pseudo-stable direct summand of M for each pseudo-stable direct summand A of $E(M)$.

Proof:

$(\Rightarrow)$ Let $N = M \cap A$, where A is a pseudo-stable direct summand of $E(M)$.Let $f: N \to M$ be an R-monomorphism. Since $E(M)$ is an injective, then there exists $g \in End_R(E(M))$ such that g is extends f Let n in N, then f(n) in M, so, since $f(n) = g(n)$ and A is pseudo-stable submodule of $E(M)$, then $f(n) \in A$. Hence, we have $f(n) \in N = M \cap A$. Thus N is a pseudo-stable submodule of M. Now, we claim that N is a direct summand of M. Since M is pseudo-extending, then there exists a direct summand D of M such that N is essential in D. Since A is a direct summand of $E(M)$, and $E(M)$ is an injective, then A is an injective[12]. Since M is essential in $E(M)$, and A is essential in A, then $N = M \cap A$ is essential in $E(M) \cap A = A$. That is N is essential in A. Hence $E(N) = E(A)$. Since $N \subseteq A$ then $E(N) = A$ and $E(N) = E(D)$. Hence $E(D) = A$. That is, $D \subseteq M \cap E(D) = M \cap A = N$. So N=D. Therefore $M \cap A = N$ is

$(\Leftarrow)$ Let K be a pseudo-stable submodule of M, then $K \oplus B$ is essential in M where B is a relative complement of K in M [2]. But M is essential in $E(M)$, then $K \oplus B$ is essential in $E(M)$. Now, since M is a fully hyperpseudo-stable, then K is a hyperpseudo stable submodule of M. That is $E(K)$ is a pseudo-stable submodule of $E(M)$. Hence by hypothesis $E(K) \cap M$ is a pseudo-stable direct summand of M. Since K is essential in $E(K)$ and M is essential in M, then $K = K \cap M$ is essential in $E(K) \cap M$. Therefore M is pseudo-extending. ■

Proposition3.1

If M is a finitely generated torsion free R-module over principal ideal domain R. Then M is pseudo-extending module

Proof:

Let A be a pseudo-stable submodule of M, and let K be a submodule of M such that $A \subseteq K \ and \ K/A$ is the torsion submodule of M/A. Since M is finitely generated, then M/K is a finitely generated [5]. then by third isomorphism theorem $M/K \cong (M/A)/(K/A)$. Since K/A is torsion submodule of M/A, then M/K is torsion free. Thus M/K is a finitely generated and torsion free R-module, so by [6] M/K is free R-module. Consider the following short exact sequence.

$$(0) \to K \overset{i_K}{\to} M \overset{f}{\to} M/K \to (0)$$

Since M/K is a free R-module, then by [7] the sequence splits. Thus K is a direct summand of M. To show that A is essential in K. Let $0 \neq y \in K \ and \ y \notin A$. Hence $y + A \neq A$. But K/A is the torsion submodule of M/A, so there exists $0 \neq r \in R$ such that $r(x + A) = A$. That is, $rx + A = A$. Since M is torsion , then $rx \neq 0$, and

$rx \in A$. Thus A is essential in K and K is a summand of M. Hence M is a pseudo-extending module ■.

Proposition3.2

Let M be a hyperpseudo-stable R-module such that for every pseudo-stable summand B of $E(M), B+M$ is a projective R-module, then M is a Ppseudo-extending module.

Proof:

Let B be a pseudo-stable summand of $E(M)$. Consider the following short exact sequences

$(0) \to B \cap M \xrightarrow{i_1} M \xrightarrow{f_1} M/(B \cap M) \to (0)$

$(0) \to B \xrightarrow{i_2} B+M \xrightarrow{f_2} (B+M)/B \to (0)$

By second isomorphism theorem, we have;$M/(B \cap M) \cong (B+M)/B$. But B is a summand of $E(M)$ and $B \subseteq B+M$, then B is a summand of B+M [11]. Thus the second sequence splits. But B+M is a projective, then $(B+M)/B$ is a projective. But, $M/(B \cap M) \cong (B+M)/B$,then $M/(B \cap M)$ is a projective. Hence the first sequence is splits. Thus $B \cap M$ is a summand of M. To prove that $B \cap M$ is a pseudo-stable summand of M, let $f: B \cap M \to M$be an R-monomorphism. Since E(M) is injective, then there exists $g \in End_R(E(M))$ such that g extends f . Let $n \in B \cap M$, then $f(n) \in M$, so since $f(n) = g(n)$ and B is a pseudo-stable submodule of E(M), then $f(n) \in B$. Hence, we have $f(n) \in B \cap M$. Thus $B \cap M$ is a pseudo-stable summand of M. Hence by Theorem2.5 M is a pseudo-extending .■

Definition 3.3

An R-module M is called a pseudo-stable uniform, if every non-zero pseudo-stable submodule of M is essential in M.

Proposition 1.2.20

If M is a pseudo-stable uniform R-module, then M is a pseudo-extending

Proof:

Let A be a pseudo-stable submodule of M, then A is essential in M. since M is a direct summand of M, then A is essential in a direct summand of M, hence M is a pseudo-extending.

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§3: Modules imply pseudo-extending modules

In this section we establish modules which imply pseudo-extending module.

Recall that an R-module M is torsion free, if $T(M) = \{m \in M : \exists r \in R, rm = 0\} = (0)$ [4]

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المقاسات الموسعة الكاذبة ومفاهيم ذات علاقة

هيبة كريم محمد علي و حيران إبراهيم فارس سعيد

الخلاصة:

قدمنا في هذا البحث مفهوم مقاس التوسع الكاذب الذي يكون تعميم لمقاسات التوسع. تم الحصول على عدد من المكافئات والخواص لهذا المفهوم. كذلك إعطاء مكافئات لهذا المفهوم في بعض أصناف المقاسات. كذلك درسنا المقاسات التي تعطي مقاسات التوسع الكاذبة.

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