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Ex-Modules and Related concepts

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Abstract: In this work , We introduce the concept of an Ex –Module as a generalization of the concept Q-Module. Many characterizations and properties of Ex-Modules are obtained .Some classes of modules which are Ex-Modules are given . We investigate conditions for Ex-Modules to be Q-Modules . Modules which are related to Ex-Modules are studied . Furthermore , characterizations of Ex-Modules in some classes of modules are obtain

Key words : Ex –Module , generalization , Q-Module, characterizations , properties

Introduction

Throughout this paper, R will denoted an associative ring with identity , and all R-modules are unitary (left) R-modules . An R-module M is said to be a Q-Module if every submodule of M is a quasi-injective[2.]. An R-module, M is called an extending if every submodule of M is an essential in a direct summand of M [4] . An R-module M is called uniform, if every non submodule of M is an essential in M, where a non-zero submodule N of M is essential in M if $N \cap K \neq (0)$ for each non zero submodule K of M, which is equivalent to say that every non-zero element $m \in M$ there exists a non-zero element r $\in R$ such that $0 \neq mr \in N[6]$. A submodule N of an R-modules M is called a fully invariant, if $f(N) \subseteq N$ for each $f \in End_R(M)[15]$.

§1: Basic properties of Ex-Module

In this section, we introduce the definition of Ex-Modules and give examples, some basic properties and characterizations of this concept.

Definition 1.1

An R-module M is called an Ex-Module**,** if every submodule of M is extending.

Example and Remarks 1.2

1. Every uniform R-module is an Ex-Module.and an R-module M is uniform if every non-zero submodule of M is an essential in M [6] .or every uniform module is an Ex-Module , but the converse is not true . Since Z_6 as Z-module , but not uniform .
2. . Every simple R-module is an Ex-Module.
3. Z_{p}^{∞} as a Z-module is an Ex-Module
4. Z_n as a Z-Module for n $>$1 is an Ex-Module.
5. . Any submodule (direct summand) of an Ex-Module is an Ex-Module.
6. Z as Z-module is an Ex-module
7. The direct sum of two distinct Ex-Modules need not to be an Ex-Module for example Z-module Z_2 and Z_4 are an Ex-Module**,** but $Z_2 \oplus Z_4$ is not an Ex-Module, since $Z_2 \oplus Z_4$ itself is not extending Z-module**.**
8. If M is an Ex-Module, then $M \oplus M$ is not necessary an Ex-Module, For example, if we take M$=Z_4$ as a Z-module, M is Ex-module, but $M \oplus M = Z_4 \oplus Z_4$ is not Ex-Module, since there exists a submodule $\{\bar{0}, \bar{2}\} \oplus Z_4$ of $Z_2 \oplus Z_4$ which is isomorphic to $Z_2 \oplus Z_4$ is not an extending Z-module.

Before we the give main result of this section ,we introduce the following lemma .

Lemma 1.3

Any fully invariant submodule of an extending module is an extending.

Proof.

let N be a fully invariant submodule of M , and A is any submodule of N, then A is a submodule of M. since M is an extending module , then, there exists a direct summand K of M such that A is essential in K. That is $M = K \oplus L$, where L is any submodule of M. Since N is a fully invariant submodule of M, then $N = (K \cap N) \oplus (L \cap N)$ .That is $K \cap N$ is a direct summand of N since A is essential in K and N is essential in N ,then $A = A \cap N$ is essential in $K \cap N$. Hence N is an extending submodule of M.

Recall that an R-module M is duo module, if every submodule of M is a fully invariant [15].

Proposition 1.4

Every duo extending module is an Ex-Module.

Proof

It follows directly by lemma 1.3

Proposition 1.5

If M is an extending module in which all its submodules are annihilator, then M is an Ex-Module.

Proof

Let K be a submodule of M. then K is an annihilator submodule of M. That is $K = ann_M(I)$ for some ideal I of R. We claim that K is a fully invariant submodule of M. Let $f \in End_R(M)$, thus $(0) = f(IK) =$ I f (K) and, hence f $(K) \subseteq ann_R(I) = K$. That is K is a fully invariant submodule of M. By lemma 1.3 , K is an extending submodule of M. Hence M is an Ex-Modul

Theorem 1.6

Let M be an R-module. Then the following statements are equivalent.

1) M is an Ex-Module.
2) M is an extending and every essential submodule of M is a fully invariant in M.
3) Every essential submodule of M is an extending.

Proof $(1) \Longrightarrow (2)$

Let N be any essential submodule of M , then N is an extending . Now , let $f \in End_R(M)$ and $0 \neq n \in N$, that is $n = 1$, $n \in N$, $f(n) = f(1 \cdot n) = 1 \cdot f(n) \in N$,because N is an essential in M . That is $f(n) \in N$. Hence $f(N) \subseteq N$. Therefore N is a fully invariant submodule in M .

$(2) \Longrightarrow (3)$

Let N be an essential submodule of M , then by hypothesis N is a fully invariant in M . Hence by lemma 1.3 N is an extending submodule of M .

$(3) \Longrightarrow (1)$

Let N be a submodule of M, then $N \oplus C$ is an essential in M where C is a relative complement of N in M[6]. Hence by hypothesis $N \oplus C$ is an extending submodule of M . which implies that N is an extending [4]. Hence M is an extending

Recall that a submodule N of an R-module M is closed in M, if N has no proper essential extension [6].

The following theorem is another characterization of an Ex-Module.

Theorem 1.7

Let M be an R-module , and N is any submodule of M, then the following statements are equivalent.

1) M is an Ex-Module.

2) Every closed submodule of N is a direct summand of N.

3) If A is a summand of the injective hull $E(N)$ of N, then $A \cap N$ is a summand of N.

Proof

$(1) \Longrightarrow (2)$

Since N $\subseteq$ M , N id extending .Hence the result follows by [16 ,prop 2.4] is hold .

$(2) \Longrightarrow (3)$ Let A is a summand of E(N), then $E(N) = A \oplus B$ where B is a submodule of $E(N)$. To prove that $A \cap N$ is closed in N. Suppose that $A \cap N$ is essential in K where K is a submodule of N, To prove that $A \cap N$ is closed in N. Suppose that $A \cap N$ essential K $\subseteq$N, so we must prove K=N($A \cap N$has no proper essential extension in N),and let $k \in$ K . Thus $k =$ a+b, where a $\in$ A , b $\in$ B . Now if a $\notin K$, then b $\neq$ 0. But N is essential in E(N) and $0 \neq b \in B \subseteq E(N)$. Therefore, there exists $r \in R$ such that $0 \neq rb \in N$. Now, $rk = ra + rb$ and, hence $ra = rb - rk \in N \cap A \subseteq K$. Thus $rb = rK - ra \in B \cap K$. But $A \cap N$ is essential in K, so $(0) = (A \cap N) \cap B$ is essential in $K \cap B$ and, hence $K \cap B = (0)$. Then $rb = 0$ which is a contradiction a $\notin K$. Thus $A \cap N$ is closed in N, and hence by hypothesis $A \cap N$ is a summand of N.

$(3) \Longrightarrow (1)$ Let N be a submodule of M, and let A be a submodule of N, then $A \oplus B$ is essential in N [6], where B is a relative complement of A in N. Since N is essential in $E(N)$, then $A \oplus B$ is essential in $E(N)$. Thus E (A $\oplus$ B) =E(A) $\oplus$ E(B). Since $E(A)$ is a summand of $E(N)$, then $E(A) \cap N$ is a summand of N. Now, $A = A \cap N$ is essential in $E(A) \cap N$ [6] and $E(A) \cap N$ is a summand of N .Hence N is an extending. Therefore M is an Ex-Module.

§ 2:Modules imply Ex-Modules

In this section we establish modules which imply Ex-Modules.

Recall that an R-module M satisfies Bares' criterion, if every submodule of M satisfies Baers̓ criterion ,where we say that a submodule N of M satisfies Baers̓ criterion, if for each R-homomorphism

$f: N \longrightarrow M$, there exists $r \text{ in } R$ such that $f(n) = rn, \forall n \in N$ [1].

Proposition 2.1

If M is an extending module which satisfies Bear's criterion, then M is an Ex-Module.

Proof

Let K be a submodule of M, then K satisfies Bears' criterion, hence K is a fully invariant submodule of M (since for each f $\in End_R(M)$ and for each k $\in$ K, $f(k)$ =rk $\in$ K, for some $r \in R$. That is $f(k) \in K$. Which implies that $f(K) \subseteq K$). By lemma 1.3 K is an extending. Hence M is an Ex-Module.

Recall that a submodule N of an R-module M is annihilator, if $N = ann_M(I)$ for some ideal I of R [15].

Proposition 2.2

If M is extending module such that every cyclic submodule of M is a fully invariant in M. Then M is an Ex-Module.

Proof.

Let K be a submodule of M., then for each $f \in End_R(M)$ and for each x$\in$ K, we have $f((x)) \subseteq (x) \subseteq K$. Thus, $f(x) \in K$. Hence $f(K) \subseteq$ K. That is K is a fully invariant in M. By lemma 1.3 K is an extending submodule of M. Therefore, M is an Ex-Module.

Proposition 2.3

If M is an extending module such that every submodule of M is closed, then M is an Ex-Module.

Proof

Let K be a submodule of M, then K is closed submodule and K is a direct summand of M, hence K is an extending [4]. Therefore M is an Ex-Module.

The following proposition shows that under a certain condition Ex-Modules and uniform modules are equivalent.

Proposition 2.4

Let M be an indecomposable R-module. Then M is uniform if and only if M is an Ex-Module.

Proof

$(\Longrightarrow)$ By examples and remarks 1.2

$(\Longleftarrow)$ directly from [16,pro 2. , p20] ,Since every Ex-Module is extending .Hence M is a uniform .

Recall that an R-module M is torsion free, if $\mathcal{T}(M) = \{m \in M: rm = 0$ for some $r \in R\} = (0)$ [6] .

Proposition 2.5

Let M be a torsion free R-module over principle ideal domain R, such that every submodule of M is a finitely generated, then M is an Ex-Module.

Proof

Let N be a submodule of M ,and let A be a submodule of N, and C be a submodule of N containing A such that $\frac{C}{A}$ is a torsion free submodule of $\frac{N}{A}$. Since N is a finitely generated, then $\frac{N}{C}$ is a finitely generated. Hence by the third isomorphism theorem $\frac{N}{C} \cong \frac{\frac{N}{A}}{\frac{C}{A}}$. But $\frac{N}{C}$ is a finitely generated and torsion free R-submodule. Then $\frac{N}{C}$ is a free [7]. Now, consider the following short exact sequence$0 \longrightarrow C \xrightarrow{i} N \xrightarrow{f} \frac{N}{C} \longrightarrow 0$ where i is the inclusion mapping and f is the natural epimorphism. Since $\frac{N}{C}$ is a free R-module, the sequence is split [7]. Thus, C is a direct summand of N .Now let $0 \neq y \in C$, and $y \notin A$, then $y + A \neq A$, but $\frac{C}{A}$ is torsion submodule of $\frac{N}{A}$, so there exists $0 \neq r \in R$, such that $ry + A = A$. But N is a torsion free, then $0 \neq ry \in A$. Thus, A is essential in C and C is a direct summand of A in N. Hence N is an extending. Therefore M is an Ex-Module.

Recall that An R-module M is π-injective , if $f(M) \subseteq M$ for every idempotent $f \in End(E(M))$ [4].equivalent , M is $\pi -$inj $\leftrightarrow$M is extending +C_3

Proposition 2.6

Let M be an R-module such that every submodule of M is a π-injective, then M is an a Ex-Module.

Proof

Since $\pi -$inj $\leftrightarrow$M is extending +C_3

Let every submodule is π-injective .we get immediately every submodule is extending . Hence M an Ex-Module Module.

Recall that An R-module M is a projective , if for each epimorphism $g : A \longrightarrow B$ (where A, B be R-modules) and for each R-homomorphism $f : M \longrightarrow B$, there exists an R-homomorphism $h: M \longrightarrow A$ such that $goh = f$ [7].

Proposition 2.7

Let M be an R-module and N is any submodule of M such that for every summand A of $E(N)$, $A + N$ is a projective, then M is an Ex-Module.

Proof

Let N be a submodule of M, and A be a summand of $E(N)$. To prove that $A \cap N$ is a summand of N. Consider the following short exact sequences.

$0 \rightarrow A \cap N \xrightarrow{i_1} N \xrightarrow{f_1} \frac{N}{A \cap N} \rightarrow 0$ ..(1)

$0 \to A \xrightarrow{i_2} A + N \xrightarrow{f_2} \frac{A+N}{A} \to 0$ ..(2)

Where i_1, i_2 are the inclusion homomorphism and f_1, f_2 are the natural epimorphism. By the second isomorphism theorem $\frac{N}{A \cap N} \cong \frac{A+N}{A}$. It is clear A is a summand of $A + N$. Thus the second sequence splits. Since $A + N$ is a projective, then $\frac{N}{A \cap N} \cong \frac{A+N}{A}$ is a projective. Hence the first sequence splits. Thus $A \cap N$ is a summand of N. Hence by theorem 1.5 M is an Ex-Module.

Recall that An R-module M is a P-Module, if every submodule of M is a pseudo-injective. [11] .

The following results show that the P-Modules implies to Ex-Modules.

Proposition 2.8

Any P-Module over a principle ideal domain is an Ex-Module.

Proof

Let N be a submodule of M, then N is a pseudo-injective, then N is a quasi-injective [14]. Hence N is an extending [8]. Therefore M is an Ex-Module.

Proposition 2.9

Any P-Module over a Dedekind domain is an Ex-Module

Proof

Let N be a submodule of M, then N is a pseudo-injective . Thus N is a quasi-injective [13], then N is an extending [8] . Hence M is an Ex-Module.

§3:Ex-Modules and Q-Modules

In this section the relation between Ex-Modules and Q-Modules are studied. Since every a quasi-injective R-module is an extending but the converse is not true [8], then every Q-Module is an Ex-Module, but the converse is not true ,since Z as Z-module is an Ex-Module but it is not a Q-Module .Thus we put a conditions for an Ex-Module to be Q-Module.

proposition 3.1

Let M be a non-singular P-Module, then M is a Q-Module if and only if M is an Ex-Module.

Proof

$(\Longrightarrow)$ Trivial.

$(\Longleftarrow)$ Let N be a submodule of M, then N is a Pseudo-injective, since M is a non-singular, then N is a non-singular [8]. To prove that N is a quasi-injective, let A be a submodule of N and $f: A \longrightarrow N$ be an R-homomorphism. Since M is an Ex-Module, then A is an extending submodule of M, then $A = B \oplus C$ where B, C are a direct summand of A such that $Kerf$ is an essential submodule in C. Since $\frac{A}{kerf}$ is embeded in N and N is a non-singular, then $\frac{A}{kerf}$ is a non-singular, so, $kerf$ is a closed submodule of A. That is $A = B \oplus kerf$. It is clear that f is restricted to B which is a monomorphism. Since N is an extending, then $N = B_1 \oplus D$ where B is an essential in B_1. Since B_1 is a pseudo-injective, then f restricted to B extended to a homomorphism $g: B_1 \longrightarrow B_1$. For any $x \in N$, we have $x = b + d$, where $b \in B_1$ and $d \in D$. Define a mapping $h: N \longrightarrow N$ by setting $h(x) = g(b)$. Then it is clear that h is an R-homomorphism of $N \longrightarrow N$ that extends f. Thus, N is a quasi-injective. Hence M is a Q-Module.

corollary 3.2

Let M be a non-singular Ex-Module, then M is a Q-Module if and only if M is a P-Module.

Proof

$(\Longrightarrow)$ See [11]

$(\Longleftarrow)$ Follows from proposition 3.1

The following result is another sufficient condition for an Ex-Module to become Q-Module.

proposition 3.3

If M is a P-Module over Noetherian ring, then M is a Q-Module if and only if M is an Ex-Module.

Proof

$(\Longrightarrow)$ Trivial.

$(\Longleftarrow)$ Let N be a submodule of M, then N is an extending submodule of M . Thus N is a direct sum of a uniform submodule of M [10]. Since M is a P-Module, then N is a pseudo-injective. But a direct summand of a pseudo-injective is a pseudo-injective, therefore N is a direct sum of a uniform pseudo-injective submodule. Hence N is a quasi-injective [13]. Thus M is a Q-Module.

Corollary 3.4

If M is an Ex-Module over a Noetherian ring, then M is a Q-Module if and only if M is a P-Module.

Proof

$(\Longrightarrow)$ See [11]

$(\Longleftarrow)$ By proposition 3.3 .

§4:Ex-Modules and multiplication modules

An R-module M is called multiplication module, if every submodule of M is of the from IM for some ideal I of R [3]. In this section we study the relation of multiplication modules with Ex-Modules We preface our section by the following theorem which gives the relation between Ex-Modules over R and Ex-Modules over S=End$_R$(M) .

Theorem 4.1

If M is a multiplication R-module, then M is an Ex-Module over R if and only if M is an Ex-module over S where $S = End_R(M)$.

Proof $(\Longrightarrow)$

Let N be S-submodule of M. It is clear that N is an R-submodule of M , so that N is extending. Hence M is an Ex-Module over S .

$(\Longleftarrow)$ Since M is a multiplication R-module, then $S = End_R(M)$ is commutative ring and each R-submodule of M is an S-submodule [5]. Let N be R-submodule of M , so N is an S-submodule of M. Since M is an Ex-Module over S, then N is an extending, so M is an Ex-module over R.

The following proposition shows that the two concepts Ex-Modules and extending modules are equivalent in the class of multiplication modules.

Proposition 4.2

If M is a multiplication R-module, then M is an Ex-Module if and only if M is an extending.

Proof

$(\Longleftarrow)$ Let N be a submodule of M. Then $N = IM$ for some ideal I of R. Let $f \in End_R(M)$, then $f(N) = f(IM) = I\,f(M) \subseteq IM = N$. That is N is a fully invariant submodule of M. Since M is an extending, then by lemma 1.3 N is an extending. Hence M is an Ex-Module.

Theorem 4.3

Let M be a multiplication module with $ann_R(M)$ is a prime ideal of R. Then M is an Ex-Module if and only if every a quasi- invertible submodule of M is an extending.

Proof

$(\Longleftarrow)$ Let N be a submodule of M, then $N \oplus K$ is essential submodule of M, where K is the relative complement of N in M. Then $N \oplus K$ is a quasi-invertible submodule of M. Since N $\oplus$ M is essential in M,N $\oplus$ M is quasi in [9,Th 3.11,P.18] which implies that $N \oplus K$ is a quasi-invertible submodule of M. Hence by [4 ,p.55]. N is an extending submodule of M.. Therefore M is an Ex-Module.

§5:characterizations of Ex-Modules in some types of modules.

In this section, we give characterizations of an Ex-Module in some types of modules.

The following theorem gives many characterization of an Ex-Module in class of non-singular modules .

Theorem 5.1

Let M be a non-singular R-module. Then the following statements are equivalent.

1.M is an Ex-Module**.**

2.Every a quasi-invertible submodule of M is an extending.

3.Every dense submodule of M is an extending.

Proof

$(1) \Rightarrow (2)$ Trivial

$(2) \Rightarrow (3)$

Let N be a dense submodule of M ,then N is an essential submodule of M [8]. We claim that N is a quasi-invertible submodule of M. Let $f \in Hom_R\left(\frac{M}{N \oplus K}, M\right), f \neq 0$. Thus there exists $x \in M$, such that $f(x+N) = m \neq 0$, where $m \in M$. Let $r \in R$ and $r \notin ann_R(M)$. Hence, $rx \notin N$. Since N is an essential submodule of M, then there exists a non-zero element $s \in R$ such that srx is a non-zero element of N. Thus $0 = f(srx+N) = sr\,f(x+N) = srm$, this implies that $sr \in ann_R(m)$. Therefore $ann_R(m)$ is an essential ideal of R. Since M is a non-singular, then $m = 0$ and, hence $f = 0$.Therefore $Hom_R\left(\frac{M}{N}, M\right) = 0$, which implies that N is a qausi-invertible submodule of M. Hence N is an extending.

$(3) \Longrightarrow (1)$ Let N be a submodule of M, then $N \oplus K$ is an essential submodule of M, where K is the a relative complement of N in M. Since M is a non-singular, then $N \oplus K$ is a dense submodule of M [8]. Hence $N \oplus K$ is an extending submodule of M. Hence N is an extending submodule of M [4].Therefore M is an Ex-Module.

Recall that the Jacobson radical of an R-module M denoted by J(M), is defined to be intersection of all maximal submodule of M .[6]

Theorem 5.2

Let M be an R-module such that $J(End(M)) = (0)$, then M is an Ex-Module if and only if M is an extending and every a quasi-invertible submodule of M is an extending.

Proof

$(\Longleftarrow)$ Let N be a submodule of M, then $N \oplus K$ is an essential submodule of M (where k is a relative complement of N in M). We claim that $N \oplus K$ is a quasi-invertible submodule of M. Let $f \in Hom_R\left(\frac{M}{N \oplus K}, M\right)$ and $f \neq 0$. Define $g = f \circ \pi$ where π:M $\to \frac{M}{N \oplus K}$ is natural homomorphism. Hence $g \subset End_R(M)$ and $g \neq 0$ and $N \oplus K \subseteq ker g$. Since $N \oplus K$ is an essential submodule of M and hence $g \in J(End_R(M))$ then $g = 0$, this implies that $f = 0$, this is a contradiction. Therefore $Hom_R\left(\frac{M}{N \oplus K}, M\right) = (0)$, and hence $N \oplus K$ is a quasi-invertible submodule of M. Thus $N \oplus K$ is an extending. Hence N is an extending [4]. Therefore M is an Ex-Modul

REFERENCES

[1]. Abbas M.S. "On fully stable modules" Ph.D. thesis, Univ. of Baghdad, 1990.
[2]. AL-Rikaby, H. K. "Q-Modules" Ph.D. thesis, Univ. of Tikrit, 2005.
[3]. Barnard, A. "Multiplication modules" J. of Algebra, 71, (1981), 174-178.
[4]. Dung, N. V., Huynh, D. V., Simith, P. F. and Wisbauer R. "Extending Modules" Pitman Research Note in Mathematics series 313, Longman, New-York, 1994.
[5]. EL-Bast, Z. A. and Simth, P. F. "Multiplication modules" comm. In Algebra, 16 (4), (1988), 755-795.
[6]. Goodearl, K. R. "Ring theory" Marcel Dekker, New-York, 1976.
[7]. Kasch, F. "Modules and Rings" Acad. Press, London, 1982.
[8]. Lam, T. Y. "Lecture on modules and ring" Springer Verlag, New-York, 1998.
[9]. Mijbass, A. S. "Quasi-Dedekind modules" Ph.D. thesis, Univ. of Baghdad, 1997.
[10]. Okada, M. "On the decomposition of extending modules" Math. Japonica, 29 NO. 6, (1984) 939-941.
[11]. Shalaby, M. A. "P-Modules" Msc. thesis, Univ. of Tikrit, 2008.
[12]. Singh, S. and Jain, S. K. "On pseudo-injective modules and self-pseudo-injective rings" J. of Math. Sci. 2, (1967), 23-31.
[13]. Singh, S. and Wason, K. "Pseudo-injective modules over commutative rings" J. of Indian, Math. Soc. 34, (1970), 61-65.
[14]. Surjeet, S. "On pseudo-injective modules" Riv. Math., Univ. Parma, 2(9), (1968), 59-65.
[15]. Weakly, W. D. "Modules whose distinct submodules are not isomorphic" Comm. Algebra, 15, (1987), 1569-1587.
[16]. S.HMohamed and B.J.Muller ,Continuous and Discrete Modules ,London Math .Soc LNS147 Cambridge Unuv . Press, Cambridge , 1990 .

المقاسات من النمط – EX و مفاهيم أخرى

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الخلاصة:قدمنا في هذا البحث مفهوم جديد سمي المقاسات من النمط – E x كتعميم للمقاسات من النمط – Q . العديد من التشخيصات والصفات لهذا المفهوم وجدت . المقاسات التي علاقة مع المقاسات من النمط – Ex درست . فضلا عن ذلك تشخيصات أخرى للمقاسات من النمط – Ex في بعض أصناف المقاسات وجدت . العلاقة بين المقاسات من النمط – E x والمقاسات من النمط – Q أعطيت .