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Intuitionistic fuzzy projective geometry

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Abstract: In this paper , we introduce a new model of intuitionistic fuzzy projective geometry . In this model points and lines play a similar role , like they do in classical projective plane . Furthermore , we will show that this new intuitionistic fuzzy projective plane is closely related to the fibred projective geometry .

Keywords: Intuitionistic , fuzzy , geometry

Introduction :

We introduced a first model of intuitionistic fuzzy projective geometry . this provides a link between the intuitionistic fuzzy versions of classical theories that are very closely related . In another intuitionistic fuzzy model of projective geometries was constructed : fibred projective planes . In this model the role of points and lines is equivalent (this is not the case in the first model), as in the classical case . Points and lines of the base geometry mostly have multiple degrees of membership .

This paper introduces a third model . We first define an intuitionistic fuzzy projective plane in which points and lines play the same role , and such that every point and every line in the base plane possess degree of membership and degree of non membership . Also we give a definition for an n- dimensional intuitionistic fuzzy projective space and we also vestigate the link between fibred and intuitionistic fuzzy projective geometry .[4]

Definition 1.1 [1]

A projective plane r is an incidence structure (P,B,I) with P a set of points , B a set of lines and I an incidence relation , such that the following axioms are satisfied:

A(a) every pair of distinct points are incident with a unique common line .

A(b) every pair of distinct lines are incident with a unique common point .

A(c) r contains a set of four points with the property that no three of them are incident

With a common line.

A closed configuration d of r is a subset of $P \cup B$ that is closed under taking intersection points of any pair of lines in d and lines spanned by any pair of distinct points of d . We denoted the line in r spanned by the points a and b by $[a,b]$.

Definition 1.2 [2]

A projective space d is an incidence structure (P,B,I) with P a set of points , B a set of lines and I an incidence relation , such that the following axioms are satisfied:

A(a) every line is incident with at least two points .

A(b) every pair of distinct points are incident with a unique common line .

A(c) given distinct points a,b,c,d,e such that $[a,b]=[a,c]\neq[a,d]=[a,e]$, there is a point $xI[b,d]\cap[c,e]$ (Pasach s axiom) .

Definition 1.3 [5]

let X be a nonempty set. An intuitionistic fuzzy set Z on X is an object having the form

$Z=\{\langle x, l(x), m(x)\rangle : x\in X\}$ where $l : X\rightarrow I$ and $m : X\rightarrow I$,denoted the membership function and the nonmembership function of Z respectively , $I=[0,1]$, and satisfy $0< l(x)+m(x)\leq 1$ for each $x\in X$.

An intuitionistic fuzzy set $Z=\{\langle x, l(x), m(x)\rangle : x\in X\}$ can be written in the form $Z=\langle x, l, m\rangle$, or simply $Z=\langle l, m\rangle$.

Let $Z=\{\langle x, l(x), m(x)\rangle : x\in X\}$ and $F=\{\langle x, d(x), g(x)\rangle : x\in X\}$ be an intuitionistic fuzzy sets on X . Then :

(a) $\overline{Z}=\{\langle x, m(x), l(x)\rangle : x\in X\}$ (the complement of Z).

(b) $Z\bigcap F=\{\langle x, l(x)\wedge d(x), m(x)\vee g(x)\rangle : x\in X\}$ (the meet of Z and F).

(c) $Z\bigcup F=\{\langle x, l(x)\vee d(x), m(x)\wedge g(x)\rangle : x\in X\}$ (the join of Z and F).

(d) $Z\subseteq F \Leftrightarrow l(x)\leq d(x) and, m(x)\geq g(x)$ for each $x\in X$.

(e) $Z=F\Leftrightarrow Z\subseteq F$ and $F\subseteq Z$.

(f) $\tilde{1}=\{\langle x,1,0\rangle : x\in X\}$, $\tilde{0}=\{\langle x,0,1\rangle : x\in X\}$

Definition 1.4 [4]

An intuitionistic fuzzy set $Z=\{\langle x, l(x), m(x)\rangle : x\in X\}$ on n -dimensional projective space S is an intuitionistic fuzzy n - dimensional projective space on S if $l(p)\geq l(q)\wedge l(r)$ and $m(p)\leq m(q)\vee m(r)$, for any three collinear points p,q,r of S we denoted $[Z,S]$.

The projective space S is called the base projective space of $[Z,S]$ if S is an intuitionistic fuzzy point , line , plane , … , we use base point , base line , base plane, … , respectively .

Definition 1.5 [3]

Consider the projective plane r = (P,B,I) . Suppose $a\in P$ and $a, b\in[0,1]$. The IF-point (a,a,b) is the following intuitionistic fuzzy set on the point set P of r :

(a,a,b) : $P\rightarrow[0,1]$

$a\mapsto a \quad a\mapsto b$

$x\mapsto 0$ if $x\in P\setminus\{a\}$

The point a is called the base point of the IF-point (a,a,b).

An IF-line (L,a,b) with base line L is defined in a similar way .

Definition 1.6 [3]

The IF-lines (L,a,b) and (M,s,w) intersect in the unique IF- point $(L\cap M, a\wedge s, b\vee w)$.

The IF-points (a,a,b) and (b,s,w) span the unique IF-line $(\langle a,b\rangle, a\wedge s, b\vee w)$.

Definition 1.7 [3]

A fibred projective plane fr on the projective plane r consist of a set fP of IF- points and a set fB of IF-lines , such that every point and line of r is base point and base line of at least one IF-point and IF-line respectively , and such that (fP, fB) satisfies the following intuitionistic fuzzified axioms of a projective plane :

F(a) every pair of IF-points wich distinct base points span a unique IF-line .

F(a) every pair of IF-lines wich distinct base lines intersect in a unique IF-point .

The projective plane r is called the base geometry of fr .

We can construct a fibred projective plane in the following way . Let $P'\subseteq P$ and $B'\subseteq B$ be such that the unique closed configuration containing $P'\cup B'$ is $P\cup B$. For each element x of $P'\cup B'$, we choose arbitrarily a nonempty subset $\sum_x$ of $[0,1]$ of which the elements are called the initial value of x , and we define a fibred projective plane fr as follows . For each $x\in P'\cup B'$ and for each $a, b\in\sum_x$, the element (a,a,b) belongs

to fr . This is step 1 of the construction . we now describe another steps .
For any pair of IF-points that we already obtained , the IF-line spanned by it also belongs to fr by definition .
Dually , for any pair of IF-lines , the intersection IF-point belongs to fr . The set of all
IF-points and of all IF-lines constructed this way in finite number of steps is readily verified to constitute a fibred projective plane .It clear that every fibred projective plane can be constructed as above . Indeed , one can always take for each element all its corresponding values as initial values .

Now suppose $\sum_x$ is a singleton for every $x \in P \cup B$. If $P' = P$ and $B' = f$, then we call the fibred projective plane mono-point-generated . If $P' = P$ and $B' = B$ then the fibred projective plane is called mono-generated . We will restrict ourselves to these two kinds of fibred projective planes .

We see that fr can be considered as an ordinary projective plane (its base plane r) where to every point and line , a set of values from $[0,1]$ are assigned . Also the intuitionistic fuzzy projective plane in definition 1.4 can be considered as an ordinary projective plane , where to every point (and only to points) one (and only one) degrees of membership and nonmembership are assigned .

Example:

Consider the classical projective plane $f = GF(2,2)$, the Fano plane . We will construct a mono-point- generated fibred projective plane with base plane f .

We label the 7 points of f as {a,b,c,d,e,f,g} and the lines as {A,B,C,D,E,F,G}, such that :
$A = \{a,b,c\}, B = \{c,d,e\}, C = \{e,f,a\}, D = \{a,g,d\}, E = \{b,g,e\}, F = \{c,g,f\},$
$G = \{b,d,f\}.$

In step 1 , we construct the IF-points $(a,0.9,0.1),(b,0.8,0.2),(c,0.7,0.2),$ $(d,0.6,0.3),(e,0.3,0.4),(f,0.4,0.4)$ and $(g,0.5,0.3)$ on the points of P , thus $(0.9,0.1),(0.8,0.2),(0.7,0.2),$ $(0.6,0.3),(0.3,0.4),(0.4,0.4)$ and $(0.5,0.3)$ are the initial values of the respectively base points a,b,c,d,e,f and g .

Following the foregoing construction , these initial values yield the following fibred projective plane:

$\sum_a = \{(0.3,0.4),(0.4,0.4),(0.5,0.3),(0.6,0.3),(0.9,0.1)\}$,
$\sum_b = \{(0.3,0.4),(0.4,0.4),(0.5,0.3),(0.6,0.3),(0.8,0.2)\}$,
$\sum_c = \{(0.3,0.4),(0.4,0.4),(0.5,0.3),(0.6,0.3),(0.7,0.2)\}$,
$\sum_d = \{(0.3,0.4),(0.4,0.4),(0.5,0.3),(0.6,0.3)\}$,
$\sum_e = \{(0.3,0.4),(0.4,0.4),(0.5,0.3)\}$,
$\sum_f = \{(0.3,0.4),(0.4,0.4),(0.5,0.3)\}$,
$\sum_g = \{(0.3,0.4),(0.4,0.4),(0.5,0.3)\}$ and
$\sum_A = \{(0.3,0.4),(0.4,0.4),(0.5,0.3),(0.6,0.3),(0.7,0.2),(0.8,0.2)\}$
$\sum_B = \{(0.3,0.4),(0.4,0.4),(0.5,0.3),(0.6,0.3)\}$,
$\sum_C = \{(0.3,0.4),(0.4,0.4),(0.5,0.3)\}$,
$\sum_D = \{(0.3,0.4),(0.4,0.4),(0.5,0.3),(0.6,0.3)\}$,
$\sum_E = \{(0.3,0.4),(0.4,0.4),(0.5,0.3)\}$,
$\sum_F = \{(0.3,0.4),(0.4,0.4),(0.5,0.3)\}$,
$\sum_G = \{(0.3,0.4),(0.4,0.4),(0.5,0.3),(0.6,0.3)\}$.

Intuitionistic fuzzy projective plane

In this section we introduce a third model of a intuitionistic fuzzy projective geometries . Like in the fibred model , it also assigns values to the lines of the base geometry . Like the model in definition 1.4 it assigns only one value to every point (and line) of the base geometry .

Definition 2.1 [4]

Suppose r is a projective plane (P,B,I). The intuitionistic fuzzy set $Z = \langle l,m \rangle$ on $P \cup B$ is a intuitionistic fuzzy projective plane on r if :

1) $l(L) \geq l(p) \wedge l(q)$ and $m(L) \leq m(p) \vee m(q)$, $\forall p,q : \langle p,q \rangle = L$

2) $l(p) \geq l(L) \wedge l(M)$ and $m(p) \leq m(L) \vee m(M)$, $\forall L,M : L \cap M = p$

Definition 2.2

Consider the fibred projective plane fr with base plane. the projective plane is $r = (P,B,I)$. Let $\sum_p$ (respectively $\sum_L$) is the set of all different degrees of membership and nonmembership of a point p (respectively line L), for all $p \in P$ and $L \in B$. Skimming fr means that for every element x of r , we only keep the highest degree of membership and lower degree of nonmembership . This results in an intuitionistic fuzzy set c on the base projective plane r , called the cream of the fibred projective plane fr , thus :

$c : P \cup B \to [0,1]$

$x \mapsto \sup \Sigma_x$ and $x \mapsto \inf \Sigma_x$

Theorem 2.3

The cream of a fibred projective plane is an intuitionistic fuzzy projective plane, and every an intuitionistic fuzzy projective plane can be considered as the cream of a fibred projective plane .

This theorem makes sure the new definition makes sense since fibred projective planes exist and intuitionistic fuzzy projective planes will also exist .

Example

By the previous theorem , we know that the example in section 1 gives rise to the following intuitionistic fuzzy projective plane If on the Fano plane f : $(a,0.9,0.1),(b,0.8,0.2),(c,0.7,0.2),(d,0.6,0.3),(e,0.5,0.3),(f,0.5,0.3)$ and $(g,0.5,0.3)$ and points $(A,0.8,0.2),(B,0.6,0.3),(C,0.5,0.3),(D,0.6,0.3),(E,0.5,0.3),(F,0.5,0.3)$ and $(G,0.6,0.3)$.

Intuitionistic fuzzy projective spaces

So far we have only considered 2-dimensional fibred and intuitionistic fuzzy projective geometries in the plane case .
We can also define n- dimensional fibred and and intuitionistic fuzzy projective geometries , with n and an arbitrary finite integer , such that the previous theorem holds in the general case . Consider the n-dimensional projective space d . Call U_i The set of all i- dimensional subspace of d , for all $i : 0 \le i \le n-1$.

Definition 3.1

Suppose d is an n- dimensional projective space , and $i \le n$. An IF-subspace (V_i, a, b) of dimension i is the following intuitionistic fuzzy set on the set U_i of d :

$(V_i, a, b) : U_i \to [0,1]$

$V_i \to a$ and $V_i \to b$

$x \to 0$ if $x \in U_i \setminus \{V_i\}$

The subspace V_i is the base subspace of (V_i, a, b) .

Definition 3.2

An n- dimensional fibred projective space fd on the n- dimensional projective space d consist of n sets of IF-object : IF-subspaces of dimension i , for $0 \le i \le n-1$.

Every subspace (of dimention i) of d is base subspace of at least one IF-subspace (of dimention i). Moreover the following axioms have to be fulfilled :

F1) the intersection of two IF-subspaces (with distinct base subspaces that are not disjoint) is agsin an IF-subspace .

F2) every two IF-subspaces (with distinct base subspaces that do not span d itself) span an IF-subspace .

For $i = 0,1,2, n-1$, the IF-subspaces of dimention i will be called IF-points , IF-lines , IF- planes and IF-hyperplanes .

The cream of an n- dimensional fibred projective space is defined in the same way as for a fibred projective plane (see definition 3.1)

Definition 3.3 [4]

Suppose d is an n- dimensional projective space as defined above . The intuitionistic fuzzy set $Z = \langle l, m \rangle$ on $\bigcup_{i=0}^{n-1} U_i$ is a intuitionistic fuzzy projective space of dimension n on d if for all subspaces $V_i, V_j, V_k, 0 \le i, j, k \le n-1$ we have :

1) $l(V_i) \ge l(V_j) \wedge l(V_k)$ and $m(V_i) \le m(V_j) \vee m(V_k)$, $\forall$ V_j and V_k , $V_j \ne V_k$ such that $V_j \cap V_k = V_i$ if $V_i \ne f$.

2) $l(V_i) \ge l(V_j) \wedge l(V_k)$ and $m(V_i) \le m(V_j) \vee m(V_k)$, $\forall$ V_j and V_k , $V_j \ne V_k$ such that $\langle V_j, V_k \rangle = V_i$ if $V_i \ne d$.

Theorem 3.4

The cream of an n- dimensional fibred projective space is an n-dimensional intuitionistic fuzzy projective space, and every n- dimensional intuitionistic fuzzy projective space can be considered as the cream of an n-dimensional a fibred projective space .

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الخلاصة

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