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z-Compact spaces

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Abstract:z-compact and z-Lindelof spaces are studied. Characterizations of z-compact spaces and z-Lindelof spaces, using multifunctions, are given.Our main results are .A space X is z-compact iff for every space Y and z-closed graph multifunction on X into Y the image of every z-closed set in X, is closed in Y. A space X is z-Lindelof iff for every P-space Y and z-closed graph multifunction on X into Y the image of every z-closed set in X is closed in Y.

Kewords: z-compact spaces, z-Lindelof spaces compact space, H-closed space, pseudocompact space , realcompact space

Introduction

z-compact and z-Lindelof space are introduced by Frolik [1] under titles quasicompact and quasi Lindelof spaces, respectively. As far as the author knows, no further study has been done about these spaces except one result (Theorem 4.6) appeared in [3]. In this paper we study some properties of z-compact and z-Lindelof spaces. We relate z-compact spaces to pseudocompact, realcompact and H-closed spaces. Then we give some characterizations of z-compact and z-Lindelof spaces. The collection of real valued continuous forms a ring denoted by C(X) [2]. Characterizations of z-compact spaces in terms of z-filters, z-ultrafilters z-ideals and maximal ideals are given, similar to compact case, where complete regularity is assumed. Here no separation property is assumed unless otherwise is stated. For definitions and notations not stated here see [2].

Preliminaries

Definition

A subset K of a space X is called z-compact relative to X iff every cove of K by z-open sets in X has a finite subcover.

Definition

A space X is z-compact iff X is z-compact relative to X

Definition

A subset A of a space X is called a zero set iff there exists a real valued function f on X such that A=f-1(0).

Definition

A subset A of a space X is called a cozero set iff it's complement in X is a zero set

Definition

A subset of a topological space X is called z-open iff it is a union of cozero sets in X.The collection of all z-open sets in X is denoted by CR(X).

Definition

A subset is z-closed iff its complement is z-open.

A point x in X is in the z-closure of a subset A in X, ($x\in clz(A)$) if each V in CR({x}) satisfies $A\cap V\neq\phi$, A is called z-closed iff clz(A)=A, and that $x\in adz(F)$ for each $F\in\Omega$ where Ω is a family of subsets of X. We write $adz\Omega=\cap\{clz(F):F\in\Omega\}$.

Definition

A multifunction α of a space X into a space Y is a set valued function on X into Y such that $\alpha(x)\neq\phi$ for every $x\in X$.

The class of all multifunctions on X into Y is denoted by m(X,Y).

Definition

A multifunction α on X into Y is called z-closed graph iff its graph $\{(x, y)\in X\times Y: y\in\alpha(x)\}$ is z-closed in $X\times Y$.

Definition

The z-cluster set $\Delta(\alpha,x)$ of α containing $x\in X$ is defined by $\cap adz\alpha(CR(\{x\}))$.

For $K\subset X$, $\Delta(\alpha,K)$ will denote $\cup\{\Delta(\alpha,x):x\in K\}$.

Some properties of z-compact spaces

The proof of the following theorem is straightforward.

Theorem

The following statements on a space X are equivalent.

(a) X is z-compact.
(b) Every family of subsets of X each is an intersection of zero sets, with the finite intersection property has a non empty intersection.
(c) Every z-filter on X is fixed.
(d) Every z-ultra filter on X is fixed.
(e) Every z-ideal in C(X) is fixed.
(f) Every maximal z-ideal in C(X) is fixed.

Theorem

A space X is z-compact iff it is pseudocompact and realcompact.

Proof

Suppose that X is z-compact. Let f: X→ R be continuous. Then f(X) is compact and so it is bounded. So that X is peseudocompact. Also every z-ultrafilter in X is fixed . In particular every real z-ultrafilter is fixed. So X is realcompact.

Conversely suppose that X is realcompact and pseudocompact. If F is a z-ultrafilter on X then F is a real z-ultrafilter and in a pseudocompact space every real z-ultrafilter is fixed [2].

The space Y

The following well-known example Ψ [2] has many nice toplogical properties. Although it is not z-compact. It is Hausdorff, completely regular, first Axiom pseudocompact and every subset of it is a Gδ .

We describe this space for the sake of completeness. Let E be a maximal family of infinite subset of sets of natural numbers N such that the intersection of any two is finite. Let Ψ={wi: i∈E } be a new set of distinct points. The topology on Ψ is defined as follows :

Every point of N is isolated and the neighborhoods of wi are sets containing wi and all but finite numbers of E. This space is completely regular pseudocompact not realcompact consequently it is not z-compact by the above theorem.

Theorem

A countable zero set in a pseudocompact space X is z-compact relative to X.

Proof

Let K be a countable zero set in X. For each x∈K take a cozero set Vx in X containing x. Then {Vx : x∈K}∪{X-K} is a countable cover of X by cozero sets. So it has a finite subcover {Vxi : i=1,2,…,n} ∪{X - K} of X. So {Vxi : i=1,2,…,n} is a finite subcover of {Vx} to K. Then K is z-compact.

Recently it has been proved that every compact topology is contained in a maximal compact topology.

The following example shows that the situation about z-compact topology is different.

Example A z-compact topology is contained in no maximal z-compact topology. Let X be the set of real numbers with the topology

τ= {V:V⊂X, 0∉V}∪{(-1,1)}∪{X}

Then every real valued continuous function on X is constant and so X is z-compact. Now for every natural number n the topology.

$$t_n = \{V : V \subset X\} \cup \left\{\left(-\frac{1}{k}, \frac{1}{k}\right) : k = 1,2,...,n\right\} \cup X,$$

is z-compact.

As every compact topology is H-closed there are z-compact topologies which are not H-closed. For example a regular spaces on which every real valued continuous function is constant, given by Hewitt [4]. So that Hewitt space is z-compact but not compact.

Lemma

Let X and Y spaces and α∈m(X,Y). Then Δ(α,x)=Πy({x}×Y}∩clz(G(α))

Where G(α)is the graph of α

Proof

Let x∈X, y∈Δ(α,x) and let w∈CR({y}). Then for every V∈CR({x}) we have W∩α(V)≠ϕ. Then (V×W)∩G(α)≠ϕ. So (x,y)∈clz(G(α)). Hence y ∈Πy({x}×Y)∩clz(G(α)). The reverse inclusion is obtained by the obvious way. [Note that V∈CR({x}), W∈CR({y}) iff V×W∈CR({(x,y)})]. The proof is complete.

Theorem

The following statements are equivalent about spaces X and Y and α∈m(X,Y).

(a) The multifunction α has a z-closed graph G(α).
(b) α(x)=Πy(({x}×Y)∩clz(G(α)) for each x∈X.
(c) Δ(α,x)=α(x).

Proof

(a) ⇒ (b) obvious.
(b) ⇒ (c) follows from the above lemma.
(c) ⇒ (a) Let (x,y)∈X×Y-cl_z(G(α)). Then y∉α(x). So y∉Δ(α,x). Consequently there are two sets V∈CR({x}) in X and W∈CR({y}) in Y such that α(V)∩W=ϕ. Hence

$(V\times W)\cap G(\alpha)=\phi$. Therefore $cl_z(G(\alpha))\subset G(\alpha)$ and $G(\alpha)$ is closed.

Characterizations of z-compactness in terms of multifunctions

We give here several characterizations of z-compact spaces. First we need the following characterization of z-compact sets relative to X. The proof is clear.

Theorem

A subset K of a space X is z-compact relative to X if and only if for each filterbase Ω on the X such that $F\cap V\neq\phi$ is satisfied for each $F\in\Omega$ and $V\in CR(K)$ we have $K\cap adz\Omega\neq\phi$.

The following result is a characterization of z-compact spaces.

Theorem

The following statements are equivalent about a space X.

(a) X is z-compact.

(b) $\Delta(\alpha,K)=ad_z\alpha(CR(K))$ for every z-closed subset K of X and $\alpha\in m(X,Y)$.

(c) $\Delta(\alpha,K)$ is z-closed in Y for each Y, $\alpha\in m(X,Y)$ and K z-closed in X.

Proof

(a) $\Rightarrow$ (b). Let X and Y be spaces and $K\subset X$ and let $\alpha\in m(X,Y)$. For each $x\in K$ we have $CR(K)\subset CR(\{x\})$ so $\alpha(CR(K))\subset\alpha(CR(\{x\}))$ and consequently $ad_z\alpha(CR(\{x\})\subset ad_z\alpha(CR(K))$.

So, $\Delta(\alpha,x)\subset ad_z\alpha(CR(K))$ for every $x\in K$.

So, $\cup\{\Delta(\alpha,x):x\in K\}\subset ad_z\alpha(CR(K))$.

Thus $\Delta(\alpha,K)\subset ad_z\alpha(CR(K))$

Now let X be z-compact and let $K\subset X$ where K is z-closed in X.

Let $z\in ad_z\alpha(CR(K))$. Let η be a local base at z Then for $W\in\eta$ and $V\in CR(K)$ in X we have $\alpha^{-1}(W)\cap V\neq\phi$. So $\alpha^{-1}(\eta)$is a filterbase on X satisfying the hypothesis of the previous theorem. This implies that $K\cap\alpha^{-1}(\eta)\neq\phi$. For each $x\in K\cap ad_z\alpha^{-1}(\eta)$ we have $V\cap\alpha^{-1}(W)\neq\phi$ and consequently $\alpha(V)\cap W\neq\phi$ for each $V\in CR(\{x\})$ in X and $W\in\eta$. Thus $z\in\Delta(\alpha,x)$ and the proof of

(a) $\Rightarrow$ b is complete.

(b) $\Rightarrow$ (c). Obvious

(c) $\Rightarrow$ (a). Let Ω be a filterbase on X . Let $y_o\notin X$ and $Y=X\cup\{y_o\}$.

Topologize Y by taking each singleton in X open and sets containg y_o to be those sets containing a member of Ω. Hence from hypothesis $\Delta(\alpha,X)$ is z-closed in Y and we see that $y_o\in cl_z(\Delta(\alpha,x)$. Thus $y_o\in\Delta(\alpha,x)$ for some $x\in X$. For such x we have $V\cap F=V\cap(F\cup\{y_o\})\neq\phi$ for each $V\in CR(\{x\})$ and $F\in\Omega$.So, $ad_z\Omega\neq\phi$. Consequently X is z-compact.

The following result is our main characterization of z-compact spaces.

Theorem

A space X is z-compact iff for every space Y and z-closed graph multifunction $\alpha\in m(X,Y)$ the image of every z-closed set in X, is closed in Y.

Proof

Direct from the previous theorem .

z-Lindelof spaces

Definition

A space X is called z-Lindelof iff every cover of X by cozero sets in X has a countable subcover.

Hewitt's example[4] is z-Lindelof but not Lindelof.

The following results about z-Lindelof spaces can be proved by the same technique of Theorems 4.1 - 4.3

Theorem

A subset K of a space X is z-Lindelof iff for each filterbase Ω on X such that $I\cap V\neq\phi$ for every countable intersection I of elements of Ω and every $V\in CR(K)$ we have $K\cap adz\Omega\neq\phi$.

Definition

A subset of a space X is called a $G\delta$ set iff it is an intersection of a countable number of open sets.

Definition

A space X is a P-space[2] iff every $G\delta$ set is open.

Theorem

The following statements about a space X are equivalent.

(a) X is z-Lindelof.

(b) $\Delta(\alpha,K)=ad_z\alpha(CR(K))$ for every P-space Y, $\alpha\in m(X,Y)$ and z-closed subset K of X.

(c) $\Delta(\alpha,K)$ is closed in Y for every P-spaces Y, $\alpha\in m(X,Y)$ and z-closed subset K of X.

Proof

Similar to the Theorem 4.2

We conclude this paper by the following main characterization of Lindelof spaces

Theorem

A space X is z-Lindelof iff for every P-space Y and z-closed graph multifunction $\alpha\in m(X,Y)$ the image of every z-closed set in X is closed in Y.

Proof

Direct from previous theorem.

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الفضاءات المتراصة – z

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الخلاصة

تم في هذا البحث دراسة الفضاءات المتراصة –z والفضاءات اللندلوفية – z . وقد وضعت بعض التمييزات بدلالة الدوال متعددة القيم . أهم النتائج .يكون الفضاء التبولوجي X متراصاً اذا وفقط اذا تحقق ما يأتي : لكل فضاء تبولوجي Y كل دالة متعددة القيم مغلقة البيان – z على X الى Y تنقل المجموعات المغلقة-z في X الى مجموعات مغلقة في Y.يكون الفضاء X لندلوفي-z اذا تحقق الشرط الاتي : لكل فضاء تبولوجي –P Y كل دالة متعددة القيم مغلقة البيان – z على X الى Y تنقل المجموعات المغلقة-z في X الى مجموعات مغلقة في Y.