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Optimized Trigonometric Spline Function with Conjugate GradientMethod for Solving FDEs

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ABSTRACT

We investigate the non-polynomial spline function to solve the fractional differential equations with the conformable conjugate gradient method. The fractional derivative was described using the Caputo fractional derivative to construct the spline scheme with polynomial fractional order. Therefore, transform the problem to an equivalent iterative linear system that can be solved by Gauss-Seidel and conjugate gradient methods. For the given spline function, error bounds were studied and a stability analysis was completed, the error estimation is also calculated as different values of (n) depend on the step size oh (h). Numerical examples with known analytical solutions are shown to verify the method's accuracy. The outcomes are in satisfactory correlation with the exact answers according to the numerical experiments. Moreover, the convergence analysis was investigated with the drive some theorems. Also, the procedure is explained in depth and supported by computational examples and the results show that the fractional spline function which interpolates data is productive and profitable in solving unique problems and compare with the exact solutions.

Introduction:

Due to its numerous applications in science and engineering, fractional calculus plays a vital role in a multitude of fields including. materials modelling[1], electromagnetism[2], processing of signals[3], ,diffusion processes[4], fluid mechanics[5], electrical engineering[6], mathematical economics[7]. A variety of techniques have been invented to solve fractional differential equations, such as:, fractional finite difference method[8], Adomain decomposition method[9], Adam-Bashforth-Multon method[10],

Homotopy analysis[11], matrix approach method for solving FDE discussed in[12], fractional explicit Adams method used in[13]Muhammad I. Bhatti and Md.Habibur Rahman used B-polynomial bases to solve FDE[14], discrete Prabhakar fractional operator studied in[15], a spectral Tau method investigated by Hari Mohan Srivastava and et al[16].

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One of the most effective procedures for solving large linear systems of equations is the conjugate gradient method, which can also be applied to nonlinear optimization. The linear conjugate gradient method was proposed in the 1950s by Hestenes and Stiefel to solve linear systems of equations with positive definite matrices as an alternative to Gauss elimination. [17]. Fletcher and Reeves discussed the non-linear conjugate gradient method in 1964.[18], Several modifications have been introduced by researchers to the CG method, such as (,[19])

Splines have numerous applications in physics and engineering and many researchers have worked on them. For instance, H. Justine and J. Sulaiman constructed a cubic non-polynomial spline for solving BVPs [20], The application of non-polynomial spline to the numerical solution for the fractional differential equation discussed in [21], a six degree spline was found to solve second order initial value problem in [22] .

In this study we consider the fractional differential equation of the form

$$D^{\gamma}y + p(x)y' + q(x)y = r(x), \quad x \in [a,b] \tag{1}$$

With boundary conditions

$$y(a) = B_1, \qquad y(b) = B_2 \tag{2}$$

Where $p(x), q(x)$ and $r(x)$ are functions of x, B_1 and B_2 are constants. Then the interval $[a,b]$ is uniformly divided into j subintervals and the length of intervals defined as $\Delta x = h = \frac{b-a}{j}, n = j-1$.

The structure of this paper is organized as follows: we set some basic definitions and outcomes of fractional calculus in section 2, construction of non-polynomial spline function is presented in section 3, error estimations and convergence analysis is in section 4, finally some numerical examples are illustrated in section 5.

2. Mathematical Preliminaries

There are various definitions of fractional derivative and Taylor's theorem, which used in this work, will be presented in this section. The most common definitions of fractional derivatives are Caputo and Riemann-Liouville definitions.

Definition 2.1. [23] The Caputo fractional derivative of order $\lambda > 0$ is defined by

$$D^{\lambda}f(t) = \frac{1}{\Gamma(m-\lambda)}\int_a^t (t-\tau)^{m-\lambda-1}\frac{d^m}{d\tau^m}f(\tau)d\tau,$$
$$m-1 < \lambda < m \in N.$$

Definition 2.2. [12] The Riemann-Lioville fractional derivative of order $\lambda > 0$ is defined by

$$D^{\lambda}f(t) = \frac{1}{\Gamma(m-\lambda)}\frac{d^m}{d\tau^m}\int_a^t (t-\tau)^{m-\lambda-1}f(\tau)d\tau,$$
$$m-1 < \lambda < m \in N.$$

Definition 2.3.[12] The Riemann-Lioville fractional integral of order $\lambda > 0$ is defined by

$$I^{\lambda}f(t) = \frac{1}{\Gamma\lambda}\int_a^t (t-\tau)^{\lambda-1}f(\tau)d\tau,$$
$$m-1 < \lambda < m \in N.$$

Definition 2.4.[14] The Caputo derivative of order λ of a polynomial function x^d is defined by $D^{\lambda}x^d = \frac{\Gamma(d+1)}{\Gamma(d-\lambda+1)}x^{d-\lambda}$.

Definition 2.5.[24] The Spectral radius $\psi(A)$ where A is an $n \times n$ matrix is given by

$\psi(A) = \max(|\lambda|)$ where λ is an eigenvalue of A.

Definition 2.6.[24] An $n \times n$ matrix A is convergence if $\psi(A) < 1$.

Definition 2.7.[25] A square matric A is called diagonally dominate if

$|a_{ii}| > \sum_{i \neq j}|a_{ij}|$.

Theorem 2.1. For any $x_0 \in R^n$ the sequence x_k generated by conjugate gradient method converges to the solution x^* in at most n steps.

Proof. See [17].

3. Analysis of Non-Polynomial Spline Function with Fractional Order

The new model of non-polynomial spline method is to create a grid with step size x_i and fractional order, and also new conditions as the follows:

$$S(x) = S_i(x), x \in [x_i, x_{i+1}], \qquad i = 0,1,2,\dots,n \tag{3}$$

Here the trigonometric spline function with fractional order by assuming

$$S_i(x) = a_i + b_i(x-x_i)^{\frac{1}{2}} + c_i(x-x_i) + d_i cosk(x-x_i) + e_i sink(x-x_i) \tag{4}$$

Where a_i, b_i, c_i, d_i and e_i are constants for, $i = 0,1,\dots,n$, and k is a free parameter.

The spline function $S(x)$ interpolates $y(x)$ depending on k. To find the value of constants in (4) we supposed the following conditions

$$S_i(x_i) = y_i, \ S_i(x_{i+1}) = y_{i+1}, \ S_i^{\left(\frac{1}{2}\right)}(x_i) = y_i^{\left(\frac{1}{2}\right)}, \quad S_i'(x_i) = D_i, \text{ and}, S_i'(x_{i+1}) = D_{i+1}. \tag{5}$$

Applying the conditions in equation (5) the value of all constants in (4) obtained as follows

$$a_i = \frac{-\sqrt{\pi}\,M_1M_3y_{i+1} + \sqrt{\pi}\,M_1M_3y_i + 2\sqrt{h}\,M_1M_3y_i^{\left(\frac{1}{2}\right)} - (\beta M_3+\alpha)D_i + (\delta M_3+\alpha)D_{i+1}}{\alpha M_1},$$

$$b_i = \frac{-\sqrt{2k\pi}\,M_1M_2y_{i+1} + \sqrt{2k\pi}\,M_1M_2y_i + 2\,M_1\left(\alpha+\sqrt{2\theta}M_2\right)y_i^{\left(\frac{1}{2}\right)} - \sqrt{2k}(\beta M_2+\alpha)D_i + \sqrt{2k}(\delta M_2+\alpha)D_{i+1}}{\sqrt{\pi}\,\alpha M_1},$$

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$c_i = \frac{-\sqrt{\pi}\, M_1 y_{i+1} + \sqrt{\pi}\, M_1 y_i + 2\sqrt{h}\, M_1 y_i^{\left(\frac{1}{2}\right)} + (\alpha-\beta) D_i + \delta D_{i+1}}{\alpha}$,

$d_i =$

$\frac{\sqrt{\pi}\, M_1 M_3 y_{i+1} - \sqrt{\pi}\, M_1 M_3 y_i - 2\sqrt{h}\, M_1 M_3 y_i^{\left(\frac{1}{2}\right)} + (\beta M_3 + \alpha) D_i - (\delta M_3 + \alpha) D_{i+1}}{\alpha M_1}$

,

$e_i = \frac{\sqrt{\pi}\, M_1 y_{i+1} - \sqrt{\pi}\, M_1 y_i - 2\sqrt{h}\, M_1 y_i^{\left(\frac{1}{2}\right)} + \beta D_i - \delta D_{i+1}}{k\alpha}$.

Such that $\theta = kh,\ M_1 = k\sin\theta,\ M_2 = \cos\theta + \sin\theta - 1,\ M_3 = \cos\theta - 1$,
$\alpha = -2\sqrt{\pi} M_3 - \sqrt{2\theta} M_2 - \sqrt{\pi} h M_1,\ \beta = -\sqrt{\pi} M_3 + \sqrt{2\theta} - \sqrt{\pi} h M_1,\ \delta = -\sqrt{\pi} M_3 + \sqrt{2\theta},\ \ i = 0,1,2,\dots,n.$
We obtain

$$S(x) = \frac{-\sqrt{\pi}\, M_1 M_3 y_{i+1} + \sqrt{\pi}\, M_1 M_3 y_i + 2\sqrt{h}\, M_1 M_3 y_i^{\left(\frac{1}{2}\right)} -}{\alpha M_1}$$
$$+ \frac{-\sqrt{2k\pi}\, M_1 M_2 y_{i+1} + \sqrt{2k\pi}\, M_1 M_2 y_i + 2\, M_1\big(\alpha + \sqrt{2\theta}l}{\sqrt{\pi}\, \alpha i}$$
$$- x_i)^{\frac{1}{2}}$$
$$+ \frac{-\sqrt{\pi}\, M_1 y_{i+1} + \sqrt{\pi}\, M_1 y_i + 2\sqrt{h}\, M_1 y_i^{\left(\frac{1}{2}\right)} + (\alpha-\beta) D}{\alpha} \tag{6}$$
$$- x_i) +$$
$$\frac{\sqrt{\pi}\, M_1 M_3 y_{i+1} - \sqrt{\pi}\, M_1 M_3 y_i - 2\sqrt{h}\, M_1 M_3 y_i^{\left(\frac{1}{2}\right)} + (\beta M}{\alpha M_1}$$
$$- x_i) +$$
$$\frac{\sqrt{\pi}\, M_1 y_{i+1} - \sqrt{\pi}\, M_1 y_i - 2\sqrt{h}\, M_1 y_i^{\left(\frac{1}{2}\right)} + \beta D_i - \delta D_{i+1}}{k\alpha} \sin k(x - x_i).$$

Now apply the fractional continuity condition of the spline function $S_i(x)$ where the splines, $S_{i-1}^{(m)}(x) = S_i^{(m)}(x),\ m = \frac{1}{2}, 1$, we obtain the following equations

$$D^{\left(\frac{1}{2}\right)} S_i(x) = D^{\left(\frac{1}{2}\right)} y_i(x) \tag{7}$$

$$s_{i-1}^{\left(\frac{1}{2}\right)}(x)$$
$$= \frac{\sqrt{k}}{2}\left(\frac{-\sqrt{2k\pi}\, M_1 M_2 y_i + \sqrt{2k\pi}\, M_1 M_2 y_{i-}}{\sqrt{\pi}\, \alpha M_1}\right.$$
$$+ \frac{\sqrt{k}}{2}\left(\frac{-\sqrt{2k}(\beta M_2 + \alpha) D_{i-1} + \sqrt{2k}(\delta M_2 + \alpha) D_i}{\sqrt{\pi}\alpha M_1}\right).$$
$$+ \frac{\sqrt{k}}{2}\left(\frac{\sqrt{\pi}\, M_1 M_3 y_i - \sqrt{\pi}\, M_1 M_3 y_{i-1} - 2\sqrt{i}}{}\right. \tag{8}$$
$$+$$
$$\frac{\sqrt{k}}{2}\left(\frac{\sqrt{\pi}\, M_1 y_i - \sqrt{\pi}\, M_1 y_{i-1} - 2\sqrt{h}\, M_1 y_{i-1}^{\left(\frac{1}{2}\right)} + \beta D_{i-1} - \delta D}{k\alpha}\right.$$
.

From here by equating equations, (7) and (8) we obtain,

$$A_1 y_i + A_2 y_{i-1} + A_3 y_{i-1}^{(\alpha)} + A_4 D_{i-1} + A_5 D_i = 2k\sqrt{\pi}\alpha M_1 y_i^{(\alpha)} \tag{9}$$

From equation (1),

$$D^{\gamma} y_i = -p_i(x) y_i' - q_i(x) y_i + r_i(x),$$
$$D^{\gamma} y_{i-1} = -p_{i-1}(x) y_{i-1}' - q_{i-1}(x) y_{i-1} + r_{i-1}(x),$$
$$, y_i' = \frac{y_{i+1} - y_{i-1}}{2h}, and \tag{10}$$
$$y_{i-1} = \frac{-y_{i+1} + 4y_i - 3y_{i-1}}{2h}$$

Substitute equation (10) in equation (9) to obtain the iterative formula as finite difference equation

$$a_i y_{i-1} + b_i y_i + c_i y_{i+1} = F_i \tag{11}$$

Then the system of linear equation is formulated from equation (11) as follows

$$Ay = F \tag{12}$$

Where

$$A = \begin{bmatrix} b_1 & c_1 & \dots & \dots & 0 & 0 \\ a_2 & b_2 & c_2 & 0 & 0 & \vdots \\ 0 & a_3 & b_3 & c_3 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & a_{n-1} & b_{n-1} & c_{n-1} \\ 0 & 0 & \cdots & 0 & a_n & b_n \end{bmatrix}, y =$$

$[y_1\ y_2\ y_3 \dots y_{n-1}\ y_n]^T$,
$F = [F - a_1 y_0\ F_2 \dots F_{n-1}\ \ F_n - c_n y_{n+1}]^T$.

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Such that

$$a_i = A_2 + \left(\frac{3p_{i-1}}{2h} - q_{i-1}\right)A_3 - \frac{3A_4 + A_5}{2h} - \frac{\sqrt{\pi}k\alpha M_1 p_i}{h},$$

$$b_i = A_1 - \frac{2A_3 p_{i-1}}{h} + \frac{2A_4}{h} + 2\sqrt{\pi}k\alpha M_1 q_i,$$

$$c_i = \frac{A_3 p_{i-1}}{2h} + \frac{A_5 - A_4}{2h} + \frac{\sqrt{\pi}k\alpha M_1 p_i}{h},$$

$$F_i = 2\sqrt{\pi}k\alpha M_1 r_i(x_i) - A_3 r_{i-1}(x_i),\ \ i = 1,2,\dots,n.$$

$A_1 = -\sqrt{2}k^{\frac{3}{2}}\pi M_1 M_2 - 4\sqrt{\pi h}kM_1^2 + 2k^{\frac{3}{2}}\pi M_1 M_3 \cos\left(\theta + \frac{\pi}{4}\right) + 2\pi\sqrt{k}M_1^2 \sin\left(\theta + \frac{\pi}{4}\right),$

$A_2 = \sqrt{2}k^{\frac{3}{2}}\pi M_1 M_2 + 4\sqrt{\pi h}kM_1^2 - 2k^{\frac{3}{2}}\pi M_1 M_3 \cos\left(\theta + \frac{\pi}{4}\right) - 2\pi\sqrt{k}M_1^2 \sin\left(\theta + \frac{\pi}{4}\right),$

$A_3 = 2k\sqrt{\pi}M_1\left(\alpha + \sqrt{2\theta}M_2\right) + 8\theta M_1^2 - 4\sqrt{\pi\theta}kM_1M_3\cos\left(\theta + \frac{\pi}{4}\right) - 2\sqrt{\theta\pi}M_1^2 \sin\left(\theta + \frac{\pi}{4}\right),$

$A_4 = -k^{\frac{3}{2}}\sqrt{2\pi}(\alpha + \beta M_2) + 4k\sqrt{h}M_1(\alpha - \beta) + 2\sqrt{\pi}k^{\frac{3}{2}}(\alpha + \beta M_3)\cos\left(\theta + \frac{\pi}{4}\right) + 2\sqrt{k\pi}M_1 \beta \sin\left(\theta + \frac{\pi}{4}\right),$

$A_5 = k^{\frac{3}{2}}\sqrt{2\pi}(\alpha + \delta M_2) + 4k\sqrt{h}M_1\delta - 2\sqrt{\pi}k^{\frac{3}{2}}(\alpha + \delta M_3)\cos\left(\theta + \frac{\pi}{4}\right) - 2\sqrt{k\pi}M_1 \delta \sin\left(\theta + \frac{\pi}{4}\right).$

Theorem 3.1: If A is diagonally dominate then for any initial vector $x^{(0)}$ the sequence generates by Gauss-Seidel method $\{x^{(i)}\}_{i=0}^{\infty}$ convergence to unique solution of $Ax = b$.

Proof: see [24].

Theorem 3.2: If A has n independent columns then A is non-singular, A^{-1} exists, and $Au = f$ has a unique solution u.

Proof: see [26] and [21].

Theorem 3.3.[23] Let$\rho \in (0,1], p \in N$, and $f(x)$ be a continuous function in $[a,b]$ satisfying the following conditions

$(i)\ D^{j\rho}f(x) \in C([a,b]) and\ D^{\rho}f(x) \in I([a,b]), \forall j = 1,2,\dots,p.$

$(ii)\ D^{(p+1)\rho}f(x)$ is continuous on $[a,b]$.

Then for each $x \in [a,b]$,

$f(x) = \sum_{j=0}^{p} \frac{D^{j\rho}f(a)(x-a)^{j\rho}}{\Gamma(j\rho+1)} + R_p(x,a)$, where

$R_p(x,a) = D^{(p+1)\rho}f(\eta)\frac{(x-a)^{(p+1)\rho}}{\Gamma((p+1)\rho+1)}, a \le \eta \le x.$

4. Error Estimations and Convergence Analysis

Theorem 4.1: Let S(t) be the unique non-polynomial fractional spline satisfying in (4), for a given function $y(x) \in C^5[a,b]$. Then the following error estimates hold: $\left|s^{(m)}(x) - f^{(m)}(x)\right| \le \frac{h^{m+\alpha}}{(2m-2)!}\|f^{(6)}\|$, where $\alpha \in R$ and

$\left|e^{(m)}(x)\right| = \left|f^{(m)}(x) - S^{(m)}(x)\right|$, and,$w = \max_{0\le x\le h}\left|f^{(2n)}(x)\right|$.

Proof: By subtracting the analytic function $y(x)$ of sufficiently high order with the spline model in equation (6) and using theorem 3.3, we obtain.

Since $D^{\frac{1}{2}}s(x)$ is Hermite interpolation polynomial of degree 3, and matching $D^{\frac{1}{2}}f(x), D^{\frac{3}{2}}f(x)$, at $x = x_j, x_{j+1}$ so for any $x \in [x_j, x_{j+1}]$, using ([27],[28]) and let $m = 3, g = f^{\left(\frac{1}{2}\right)}$ and $p_3 = s^{\left(\frac{3}{2}\right)}(x)$, form the non-polynomial spline function in equation (4), with known constraint conditions; we get:

$\left|D^{\frac{1}{2}}s(x) - D^{\frac{1}{2}}f(x)\right| \le \frac{h^5}{4!}\left|D^{(6)}f(x)\right|$, also if we put $g = D^{\frac{3}{2}}f(x)$ and $P_3 = D^{\frac{3}{2}}s(x)$, we get

$\left|s^{\left(\frac{3}{2}\right)}(x) - f^{\left(\frac{3}{2}\right)}(x)\right| \le \frac{h^3}{3!}\left|D^{(6)}f(x)\right|$, then

$|s(x) - s(0) + f(0) - f(x)| \le \frac{h^6}{6!}\ \| f^{(6)}(x) \|$

Since $s(0) = f(0)$ and $x \in [0,1]$ then the last equation becomes

$\left|s^{\left(\frac{5}{2}\right)}(x) - f^{\left(\frac{5}{2}\right)}(x)\right| \le \frac{h^{\frac{7}{2}}}{2!}\ \| f^{(6)}(x) \|$,

and since $f^{(p)}(0) = 0, p = 1,2$, also using [23], clearly find the error estimation as follows following:

i. $let\ \zeta = 0, m = 4$, then$|e(x)| \le \frac{h^8}{8!}\left\|f^{(8)}(x)\right\|_{\infty}$.
ii. $let\ \zeta = 1, m = 4$, then$\left|e^{\left(\frac{1}{2}\right)}(x)\right| \le \frac{h^7}{6!}\left\|f^{(8)}(x)\right\|_{\infty}$.
iii. $let\ \zeta = 1, m = 4$, then $|e'(x)| \le \frac{h^6}{2!4!}\left\|f^{(8)}(x)\right\|_{\infty}$.
iv. $let\ \zeta = \frac{3}{2}, m = 4$, then $\left|e^{\left(\frac{3}{2}\right)}(x)\right| \le \frac{h^5}{3!2!}\left\|f^{(8)}(x)\right\|_{\infty}$.
v. $let\ \zeta = 2, m = 4$, then $\left|e^{(2)}(x)\right| \le \frac{h^4}{4!}\left\|f^{(8)}(x)\right\|_{\infty}$.

5. Numerical experiments

In this section the method applied to solve two numerical examples of boundary fractional differential equations with constant coefficients, the result compared with the exact analytical solution to show the efficiency of the method. The computational programs were written in MatLab. Here the algorithms of the Gauss-Seidel and the conjugate gradient methods are presented.

Algorithm 5.1

Suppose that we have the linear system (12) where A is symmetric positive definite matrix for Gauss-Seidel method first decompose matrix A as $A = D + L + U$ such that D is diagonal matrix, L is lower matrix and U is upper matrix. Then the Gauss-Seidel algorithm can be written as:

Start with initial vector $y^{(0)}$.

$y^{(i+1)} = -(D+L)^{-1}Uy^{(i)} + (D+L)^{-1}F\,, i = 0,1,2, \dots$.

Algorithm 5.2 (conjugate gradient method)

Chose $y_0 \in R^n$ and put, $d_0 = r_0 = F - Ay_0$.

For $k = 0,1,2, \dots$

If $d_k = 0, stop\ y_k\ is\ solution\ of\ Ay = F$.

Otherwise Compute

$$\alpha_k = \frac{r_k^t r_k}{d_k^t A d_k}, \quad y_{k+1} = y_k + \alpha_k d_k,$$

$$r_{k+1} = r_k - \alpha_k A d_k, \quad \beta_k = \frac{r_{k+1}^t r_{k+1}}{r_k^t r_k},$$

$$d_{k+1} = r_{k+1} + \beta_k d_k$$

Example 5.1. Consider the fractional differential equation

$$y' + y^{\alpha} + y = 2sin2x + 3cos2x\,, \quad x \in [0,\pi], \mathrm{y}(0) = \mathrm{y}(\pi) = 0 \qquad (13)$$

The exact solution of (13) when $\alpha = \frac{1}{2}$ is, $y = sin2x$.

Table 1 shows the number of iterations with different value of j using Gauss-Seidel and Conjugate gradient methods.

Table 1. *Iteration numbers of example 5.1.*

Number of iterations			
j	64	128	256
GS	161	687	6962
CG	63	127	255

Table 2 *Exact , approximate, and absolute error of example 5.1.*

x	Exact solution	Approximate solution	Absolute error
$\frac{\pi}{64}$	0.098	0.0973670	6.5×10^{-4}
$\frac{\pi}{8}$	0.7071	0.7031279	3.9×10^{-3}
$\frac{\pi}{4}$	1	0.9957816	4.2×10^{-3}
$\frac{3\pi}{8}$	0.7071	0.7062436	8.6×10^{-4}
$\frac{\pi}{2}$	0	0.0038903	3.8×10^{-3}

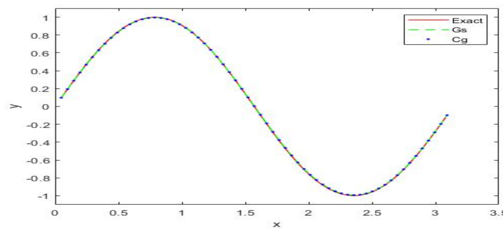


Figure 1 Exact and approximate solution of example 5.1 with $h = \frac{\pi}{64}$

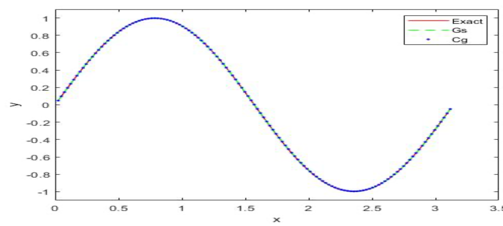


Figure 2 Exact and approximate solution of example 5.1 with $\mathrm{h} = \frac{\pi}{128}$

Example 5.2. A boundary value problem of FDE

$$y' + y = x^2 + x + \frac{8}{3}\sqrt{\frac{x^3}{\pi}} - 2\sqrt{\frac{x}{\pi}} - 1 + y^{\alpha}, x \in [0,1], y(0) = 0, and\ y(1) = 0.$$

The exact solution with $\alpha = \frac{1}{2}$ is given as, $y = x^2 - x$.

The iteration numbers is present in Table 3

Table 3 *iteration numbers of example 5.2.*

Number of iterations			
j	64	128	256
GS	220	1607	7838
CG	63	127	255

Table 4 *Exact, approximate, and absolute error of example 5.2.*

x	Exact solution	Approximate solution	Absolute error
$\frac{1}{64}$	−0.01538	−0.0134754	1.9×10^{-3}
$\frac{1}{8}$	−0.1093	−0.0982452	1.1×10^{-2}
$\frac{1}{4}$	−0.1875	−0.1772538	1.7×10^{-2}

$\frac{3}{8}$	-0.2343	-0.23071	3.6×10^{-3}
$\frac{1}{2}$	-0.25	-0.254126	4.1×10^{-3}

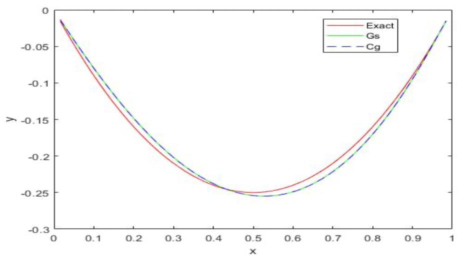
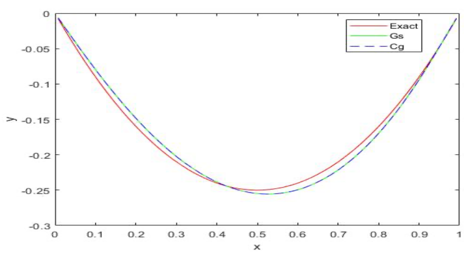


Figure 3 exact and approximate solution of example 5. 2 h=$\frac{1}{128}$



. Figure 4 exact and approximate solution of example 5.2 h=$\frac{1}{256}$.

Example 5.3.[29] Consider the fractional differential equation

$$D^{\alpha}y(x) = -y(x) + x^4 - \frac{1}{2}x^3 - \frac{3}{\Gamma(4-\alpha)}x^{3-\alpha} + \frac{4}{\Gamma(5-\alpha)}x^{4-\alpha}, 0 < \alpha < 1 .$$

With boundary conditions, $y(0) = 0, y(1) = \frac{1}{2}, and\ x \in [0,1]$.

The exact solution with $\alpha = \frac{1}{2}$ is , $y = x^4 - \frac{1}{2}x^3$.

Table 5 .Iteration numbers of example 5.3

Number of iterations	
J	1000
CG	999
GS	19710

Table 6.*Exact, approximate, and absolute error of example 5.3.*

x	Exact solution	Approximate solution	Absolute error
0.1	-0.0004	-0.00011694	2.8×10^{-4}
0.2	-0.0024	-0.000194	2.2×10^{-3}
0.3	-0.0054	-0.00003	5.4×10^{-3}
0.4	-0.0064	0.00140836	7.8×10^{-3}
0.5	0	0.00647904	6.4×10^{-3}
0.6	0.0216	0.02230459	7.0×10^{-4}
0.7	0.0686	0.06383659	4.7×10^{-3}
0.8	0.1536	0.15236168	1.2×10^{-3}
0.9	0.2916	0.30210056	1.05×10^{-2}

6. Conclusion

This paper developed the trigonometric spline method for solving FDE and conformable with conjugate gradient methods. The findings with the non-polynomial quartic spline functions are really quite interesting. In approximating functions, the non-polynomial spline and conjugate gradient approaches are more adaptive, as seen in the numerical examples. The graphs comparing exact and approximate solutions for numerical examples demonstrate our method's superiority.

7. References

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الدالة سبلاين المثلثية المحسنة مع طريقة التدرج المقترن لحل FDEs

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الخلاصة:

نحن نبحث في دالة سبلاين غير متعددة الحدود لحل المعادلات التفاضلية الكسرية باستخدام طريقة التدرج المترافق المطابق. تم وصف المشتق الكسري واستخدام التكامل الجزئي والمشتق(Caputo) يجب أن يبني الاستيفاء الخطي باستخدام معاملات متعددة الحدود الكسري. هذا ، بالتالي ، يحول المشكلة إلى نظام خطي تكراري مكافئ يمكن حله بواسطة (Gauss–Seidel) وطرق التدرج المترافق. بالنسبة لوظيفة الشريحة المحددة ، تمت دراسة حدود الخطأ واكتمل تحليل الثبات ، ويتم حساب تقدير الخطأ أيضًا نظرًا لأن قيم مختلفة لـ (n) تعتمد على حجم الخطوة (h). تم عرض أمثلة عددية مع حلول تحليلية معروفة للتحقق من دقة الطريقة. كانت النتائج في علاقة مرضية مع الإجابات الدقيقة وفقًا للتجارب العددية. علاوة على ذلك ، تم التحقيق في تحليل التقارب مع اشتقاق بعض النظريات . أيضًا ، يتم شرح الإجراء بعمق ودعمه بأمثلة حسابية وتظهر النتائج أن وظيفة الشريحة الجزئية التي تقحم البيانات منتجة ومربحة في حل المشكلات الفريدة ومقارنتها بالحلول الدقيقة.