J. of university of anbar for pure science : Vol.3: No.2 : 2009

ISSN: 1991-8941

On the analytical part of harmonic univalent functions defined by generalized SA˘LA˘GEAN Drivatives

ABDUL- RAHMAN S. JUMA   S. R. KULKARNI****
*** university of Anbar - College of Education**
****Fergusson College, Pune- 411004- India .**

Abstract: In the present paper and by making use the generalized S˘al˘agean derivatives we have introduce and study a class of analytic function and prove the coefficient conditions, distortion bound, fractional integral operator, convex combination, and radius of convexity for the analytic part of the harmonic starlike functions of order α.

Key Words : Starlike function, Harmonic function, S˘al˘agean derivative, Distortion bounds, Fractional integral operator.

Introduction

Let H denote the family of all complex-valued, harmonic orientation - preserving, univalent function f in the open unit disk U = {z : |z| < 1} for which f(0) = fz(0)−1 = 0.Each f ∈ H can be expressed as f = h + g, where h and g are the analytic and the co-analytic part of f, respectively. Then for f = h + g ∈ H we can write the analytic

functions h and g as h(z) = z + $\sum_{n=2}^{\infty} a_n z^n$, g(z)= z+ $\sum_{n=1}^{\infty} b_n z^n$, $|b_1| < 1.$ (1)

Firstly, Clunie and Sheil-Small [2] studied the class H together with some geometric subclasses and obtained some coefficient bounds. Since then, there has been severalarticles related to H and it is subclasses.The differential operator Dk was introduced by [6],and generalized by [1]. Jahangiri and et. al. [5] defined the modified S˘al˘agean operator of f as

Dkf(z) = Dkh(z) + (−1)kDkg(z), (2)

where

$D^k h(z)$= z+ $\sum_{n=2}^{\infty}(n)^k a_n z^n$ and $D^k g(z)$= $\sum_{n=1}^{\infty}(n)^k b_n z^n$, $k \in N_0 = \{0,1,2,....\}$.

Here, we define the modified generalized S˘al˘agean operator of f as

$$D_l^k f(z) = D_l^k h(z) + (-1)^k \overline{D_l^k g(z)},$$ (3)

where

$D_l^k h(z)$= z+ $\sum_{n=2}^{\infty}(1+(n-1)l)^k a_n z^n$ and $D_l^k g(z) = \sum_{n=1}^{\infty}(1+(n-1)l)^k b_n z^n$.

Also, Jahangiri [4] defined the class RAH (α) consisting of functions f = h + g such that

$$h(z) = z - \sum_{n=2}^{\infty}|a_n| z^n \ , \quad g(z) = \sum_{n=2}^{\infty}|b_n| z^n,$$ (4)

which satisfy the condition

$$\frac{\partial}{\partial q}(\arg f(re^{iq})) \ge a, 0 \le a < 1, |z| = r < 1,$$

and he proved in [4] that if $f = h + \overline{g}$ is given by (1) and if

$$\sum\left(\frac{n-a}{1-a}|a_n| + \frac{n+a}{1-a}|b_n|\right) \le 2,$$

$$0 \le a < 1, a_1 = 1,$$ (5)

then f is harmonic, univalent, and starlike of order α in U. This condition is proved to be also necessary if f ∈ RAH(α). Now, for λ ≥ 0, α ≥ 0, k $\in N_0 = \{01,2,3,...\}$ and γ ∈ C, we let RAH(k, λ, γ, α) denote class of harmonic functions f, if the analytic functions h and g satisfy the condition

$$\mathrm{Re}\{az(D_l^k h(z))'' + \frac{D_l^k g(z)}{z}\} > 1 - |g|.$$ (6)

Note that the class RAH (0; 0, γ, α) was studied by B. A. Frasin [3].

Coefficient Conditions

First we state and prove a sufficient coefficient condition for the class RAH(k, λ, γ, α).

Theorem 1 : Let f = h + $\overline{g}$ with h and g are given by (4). If f ∈ RAH(k, λ, γ, α), then

$$\sum_{n=2}^{\infty}(1+(n-1)l)^k\left[an(n-1)|a_n|-\frac{1-3a}{n+a}\right]\le|g|, \qquad (7)$$

where $a_1=b_1=0$, 0 ≤ α < $\frac{1}{3}$, λ ≥ 0, $k\in N_0=\{0,1,2,....\}$. and γ ∈ C. The result is sharp.

Proof : Let f(z) ∈ RAH (k, λ, γ, α), then by (6) we have

$$\text{Re}\left\{-\sum_{n=2}^{\infty}(1+(n-1)l)^k an(n-1)|a_n|z^{n+1}+1+\sum_{n=2}^{\infty}(1+(n-1)l)^k|b_n|z^{n-1}\right\}>1-|g|.$$

By choosing z to be real and let z → 1−, we get

$$1-\left(\sum_{n=2}^{\infty}(1+(n-1)l)^k an(n-1)|a_n|-\sum_{n=2}^{\infty}(1+(n-1)l)^k|b_n|\right)\ge1-|g|,$$

or,

$$\sum_{n=2}^{\infty}(1+(n-1)l)^k(an(n-1)|a_n|-|b_n|)\le|g|. \qquad (8)$$

Since f(z) ∈ RAH (α), we have

$$\sum_{n=2}^{\infty}\left(\frac{n+a}{1-a}|b_n|\right)\le2,$$ so,

$$|b_n|\le\frac{1-3a}{n+a}(n\ge2), \qquad (9)$$

and therefore,

$$(1+(n-1)l)^k|b_n|\le(1+(n-1)l)^k\frac{1-3a}{n+a}.$$

Then

$$\sum_{n=2}^{\infty}(1+(n-1)l)^k\left[an(n-1)|a_n|-\frac{1-3a}{n+a}\right]\le|g|.$$

This completes the proof.

Corollary 1 : Let f = h + $\overline{g}$ with h and g given by (4). If f ∈ RAH (k, λ, γ, α), then

$$|a_n|\le\frac{|g|(n+a)+(1-3a)(1+(n-1)l)^k}{an(n-1)(n+a)(1+(n-1)l)^k},$$

$$n\ge2,\ l\ge0, k\in N_0, 0\le a<\frac{1}{3} and g\in C. \qquad (10)$$

The result is sharp for the functions

$$h(z)=z-\frac{|l|(n+a)+(1-3a)(1+(n-1)l)^k}{an(n-1)(n+a)(1+(n-1)l)^k}z^n,$$

$(n\ge2)$ (11)

and

$$g(z)=z+\frac{(1-3a)(1+(n-1)l)^k}{n+a}z^n,$$

$(n\ge2).$ (12)

Theorem 2 : Let *f* = *h* + $\overline{g}$ with *h* and *g* given by (4). If |*γ*1| ≤ |*γ*2|, then *RAH*(*k*, *λ*, *γ*1, *α*) ⊂ *RAH*(*k*, *λ*, *γ*2, *α*), where 0 ≤ *α* ≤ $\frac{1}{3}$, *λ* ≥ 0 and *k* ∈ N_0.

Proof : Let *f*(*z*)∈ *RAH*(*k*; *λ*, *γ*1, *α*). Then

$$\sum_{n=2}^{\infty}(1+(n-1)l)^k\left[an(n-1)|a_n|-\frac{1-3a}{n+a}\right]\le|g1|\le|g2|.$$

This completes the proof.

Theorem 3 : Let *f* = *h* + $\overline{g}$ with *h* and *g* given by (4). If *f* ∈ *RAH* (*k*, *λ*, *γ*, *α*), then for |*z*| = *r* < 1, we have

$$r-\frac{|g|(2+a)+(1-3a)(1+l)^k}{2a(2+a)(1+l)^k}r^2\le|D_l^k h(z)|\le r+\frac{|g|(2+a)+(1-3a)(1+l)^k}{2a(a+2)(1+l)^k}r^2 \qquad (13)$$

and

$$1-\frac{|g|(2+a)+(1-3a)(1+l)^k}{2a(2+a)(1+l)^k}r\le|(D_l^k h(z))'|\le1+\frac{|g|(2+a)+(1-3a)(1+l)^k}{2a(a+2)(1+l)^k}r.$$

The results are sharp.

Proof : Let f(z) ∈ RAH (k, λ, γ, α), then from (8), we have

$$2(1+l)^k a\sum_{n=2}^{\infty}|a_n|-(1+l)^k\sum_{n=2}^{\infty}|b_n|\le|g|$$ for |z| = r < 1.

By assumption we have f(z) ∈ RAH (α), then we get

$$\sum_{n=2}^{\infty}|b_n|\le\frac{1-3a}{2+a}.$$

Therefore we conclude that

$$\sum_{n=2}^{\infty}|a_n|<\frac{|g|(2+a)+(1-3a)(1+l)^k}{2a(a+2)(1+l)^k}.$$

Then

$$|D_l^k h(z)|\ge r-\sum_{n=2}^{\infty}|a_n||r|^n\ge r-r^2\sum_{n=2}^{\infty}|a_n|$$

$$\ge r-\frac{|g|(2+a)+(1-3a)(1+l)^k}{2a(2+a)(1+l)^k}r^2$$, and

$$|D_l^k h(z)|\le r+\frac{|g|(2+a)+(1-3a)(1+l)^k}{2a(2+a)(1+l)^k}r^2.$$

Furthermore, we have from (8)

$$a(1+l)^k\sum_{n=2}^{\infty}n|a_n|-(1+l)^k\sum_{n=2}^{\infty}|b_n|\le|g|,$$

therefore, we obtain

$$\sum_{n=2}^{\infty}n|a_n|\le\frac{|g|(2+a)+(1-3a)(1+l)^k}{2a(2+a)(1+l)^k}. \qquad (14)$$

Thus,

$$|(D_l^k h(z))'|\le1+|r|\sum_{n=2}^{\infty}n|a_n|\le1+\frac{|g|(2+a)+(1-3a)(1+l)^k}{2a(2+a)(1+l)^k}r,$$ and

$$|(D_l^k h(z))'| \geq 1-\frac{|g|(2+a)+(1-3a)(1+l)^k}{2a(2+a)(1+l)^k}r.$$

For sharpness we consider the functions h(z) and g(z) given by

$$h(z)=z-\frac{|g|(2+a)+(1-3a)(1+l)^k}{2a(2+a)(1+l)^k}z^2 \quad (15)$$

and

$$g(z)=z+\frac{(1-3a)(1+l)^k}{a+2}z^2. \quad (16)$$

Fractional Integral Operator

Definition [7] : For real number β > 0, μ and δ, the fractional integral operator $I_{0,z}^{b,m,d}$ is defined by

$$I_{0,z}^{b,m,d}h(z)=\frac{z^{-b-m}}{\Gamma(b)}\int_0^z (z-t)^{b-1}F(b+m,-d;b;1-\frac{t}{z})h(t)dt,$$

where the function h(z) is analytic in a simply connected region of the z-plane containing the origin with the order h(z) = $O(|z|^e)$, (z → 0) with _ > max{0, μ − δ} − 1, and F(a, b; c; z) is the Gauss hypergeometric function defined by F(a, b; c; z)=$\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n(1)_n}z^n$, where $(n)_n$ is the Pochhammer symbol defined by

$$(n)_n=\frac{\Gamma(n+n)}{\Gamma(n)}\begin{cases}1 & n=0\\ n(n+1)\ldots\ldots(n+n-1) & n\in N\end{cases}$$

and the multiplicity of (z − t)β−1 is removed by requiring log(z − t) to be real when z-t>0.

We shall need the following result due to Srivastava et. al. [7] to prove our result.

Lemma : If β > 0 and n > μ − δ − 1, then

$$I_{0,z}^{b,m,d}\ z^n=\frac{\Gamma(n+1)\Gamma(n-m+d+1)}{\Gamma(n-m+1)\Gamma(n+b+d+1)}z^{n-m}.$$

Theorem 4 : Let β > 0, μ > 2, β + δ > −2, μ −δ > 2 and $\frac{m(b+d)}{b}\leq 3$, and let f = h + $\overline{g}$ with h and g are given by (4). If f(z) ∈ RAH (k, λ, γ, α), then

$$|I_{0,z}^{b,m,d}D_l^k h(z)|\geq\frac{\Gamma(2-m+d)|z|^{1-m}}{\Gamma(2-m)\Gamma(2+b+d)}\left\{1-\frac{[(2+a)|g|+(1-3a)(1+l)^k](2-m+d)}{a(a+2)(1+l)^k(2+b+d)}|z|\right\} \quad (17)$$

and

$$|I_{0,z}^{b,m,d}D_l^k h(z)|\leq\frac{\Gamma(2-m+d)|z|^{1-m}}{\Gamma(2-m)\Gamma(2+b+d)}\left\{1+\frac{[(2+a)|g|+(1-3a)(1+l)^k](2-m+d)}{a(a+2)(1+l)^k(2+b+d)}|z|\right\}$$

for $z\in U_0$, where

$$U_0=\begin{cases}U & m\leq 1\\ U-\{0\} & m>1\end{cases} \quad (18)$$

Proof : Putting

$$H(z)=\frac{\Gamma(2-m)\Gamma(2+d+b)}{\Gamma(2-m+d)}z^m I_{0,z}^{b,m,d}h(z),$$

and from above lemma, we have

$$H(z)=\frac{\Gamma(2-m)\Gamma(2+d+b)}{\Gamma(2-m+d)}z^m\times\left[\frac{\Gamma(2-m+d)}{\Gamma(2-m)\Gamma(2+b+d)}z^{1-m}-\sum_{n=2}^{\infty}\frac{\Gamma(n+1)\Gamma(n-m+d+1)}{\Gamma(n-m+1)\Gamma(n+b+d=1)}\times|a_n|z^{n-m}\right],z\in U_0$$

$$=z+\sum_{n=2}^{\infty}y(n)|a_n|z^n,$$ where

$$y(n)=\frac{(2-m+d)_{n-1}(1)_n}{(2-m)_{n-1}(2+b+d)_{n-1}},n\geq 2.$$

It is clear that ψ(n) is non-increasing for n ≥ 2, and we have

$$0<y(n)\leq y(2)=\frac{2(2-m+d)}{(2-m)(2+b+d)}.$$

Thus, by (8) and the last expression, we have

$$|H(z)|\geq|z|-y(2)|z|^2\sum_{n=2}^{\infty}|a_n|$$

$$\geq|z|-\frac{[|g|(2+a)+(1-3a)(1+l)^k](2-m+d)}{a(2+a)(1+l)^k(2+b+d)}|z|^2,$$

and

$$|H(z)|\leq|z|+\frac{[|g|(2+a)+(1-3a)(1+l)^k](2-m+d)}{a(2+a)(1+l)^k(2+b+d)}|z|^2$$

for $z_0\in U_0$, which was defined by (18).

Some Properties of the Class RAH(k, λ, γ, α)

Theorem 5 : The class RAH (k, λ, γ, α) is closed under convex combination.

Proof : For i = 1, 2, · · ·, let fi(z) ∈ RAH (k, λ, γ, α), where fi is given by

$$f_i(z)=z-\sum_{n=2}^{\infty}|a_{n,i}|z^n+\sum_{n=2}^{\infty}|b_{n,i}|z^n,$$

then by Theorem 1 we have

$$\sum_{n=2}^{\infty}(1+(n-1)l)^k\left[an(n-1)|a_{n,i}|-\frac{1-3a}{n+a}\right]\leq|g|.$$

The convex combination of fi for 0 ≤ ti ≤ 1 and $\sum_{i=1}^{\infty}t_i=1$ can be written as

$$\sum_{n=2}^{\infty}t_i f_i(z)=z-\sum_{n=2}^{\infty}\left(\sum_{i=1}^{\infty}t_i|a_{n,i}|\right)z^n+\sum_{n=2}^{\infty}\left(\sum_{i=1}^{\infty}t_i|b_{n,i}|\right)z^n.$$

Therefore, and from (9), we have

$$\sum_{n=2}^{\infty}(1+(n-1)l)^k\left[an(n-1)\left(|\sum_{i=1}^{\infty}t_i|a_{n,i}||-|\sum_{i=1}^{\infty}t_i\frac{1-3a}{n+a}|\right]=$$

$$\sum_{i=1}^{\infty}t_i\{\sum_{n=2}^{\infty}(1+(n-1)l)^k\left[an(n-1)|a_{n,i}|-\frac{1-3a}{n+a}\right]\}\leq\sum_{i=1}^{\infty}t_i|g|=|g|,$$

and so $\sum_{i=1}^{\infty} t_i f_i \in$ RAH (k, λ, γ, α).

Theorem 6 : Let f = h + $\overline{g}$ with h and g are given by (4). If f ∈ RAH (k, λ, γ, α), then h(z) is starlike of order σ(0 ≤ σ < 1) in |z| < r1, where

r1 $= \inf_n \left[\frac{a(2+a)(1+l)^k (1-s)n}{[(2+a)|g|+(1-3a)(1+l)^k](n-s)} \right]^{\frac{1}{n-1}}, n \geq 2$ (19)

The result is sharp for h(z) given by (15).

Proof : Suppose f ∈ RAH (k, λ, γ, α). We must show that $\left|\frac{zh'(z)}{h(z)} - 1\right| \leq 1-s$, for $|z| < r_1$, we have

$$\left|\frac{zh'(z)}{h(z)} - 1\right| \leq \frac{\sum_{n=2}^{\infty}(n-1)|a_n||z|^{n-1}}{1-\sum_{n=2}^{\infty}|a_n||z|^{n-1}}.$$

Therefore, $\left|\frac{zh'(z)}{h(z)} - 1\right| \leq 1-s$ if $\sum_{n=2}^{\infty}\frac{(n-s)}{(1-s)}|a_n||z|^{n-1} \leq 1.$

Now by using (14), the last inequality will be true if

$$\left(\frac{n-s}{1-s}\right)|z|^{n-1} \leq \frac{na(2+a)(1+l)^k}{(2+a)|g|+(1-3a)(1+l)^k},$$

or equivalently, if

$$|z| \leq \left[\frac{an(2+a)(1+l)^k(1-s)}{[(2+a)|g|+(1-3a)(1+l)^k](n-s)}\right]^{\frac{1}{n-1}}, n \geq 2.$$

Corollary 2 : Let $f = h + \overline{g}$ with h and g are given by (4). If $f \in RA_H(k, \lambda, \gamma, \alpha)$, then h(z) is convex of order σ(0 ≤ σ < 1) in |z| < r_2, where

$$r_2 = \inf_n \left[\frac{a(2+a)(1+l)^k(1-s)}{[(2+a)|g|+(1-3a)(1+l)^k](n-s)}\right]^{\frac{1}{n-1}}, n \geq 2$$

The result is sharp for h(z) given by (15).

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حول الجزء التحليلي للدوال احادية التكافؤ التوافقيه المعرفه بتعميم مشتقات سلاجين

عبدالرحمن سلمان جمعه **اس. أر. كولكارني**

E-mail : kulkarni_ferg@yahoo.com

الخلاصه

في هذا البحث وباستخدام تعميم مشتقات سلاجين قمنا بتقديم ودراسة صف للداله التحليليه وبرهنة شروط المعـــاملات ، ،مؤثر التكامل الكسري ،نصف قطر التحدب للجزء التحليلي للدوال النجميه التوافقيه من الرتبه α.