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| D:\مجلة\last\شعار المجلة.jpg **Solving system of Euler's equations using Runge –Kutta methods**  **. Aseel A Hussain and Athraa N Albukhuttar**  Department of Mahematics, College of Educaion for Girls, Universiy of Al-Kufa, Najaf – Iraq | | |
| **ARTICLE INFO** | |  | **ABSTRACT** | |
| Received: 31 / 05 /2023  Accepted: 11 / 06/ 2023  Available online: 05 / 12 / 2023   |  | | --- | | DOI: 000000000000000000000 | | |  | In this paper, linear systems with variable coefficients (Euler's equations) were solved using one of the numerical methods that are subject to initial conditions defined over a given period of time .The explicit Rung-Kutta method is the fastest and most common numerical method starting with an initial value, the Rung-Kutta second order and Rung-Kutta fourth order. Analytical solutions of systems (systems with variable coefficients and systems with constant coefficients) were compared with the results of approximate solutions of the numerical method (Rung-Kutta second order And fourth order) and find out the accuracy of the results obtained for this approximate method after applying the Rung-Kutta algorithms performed with the Matlab program and finding the ratio of relative error between the exact and approximate solutions of the numerical method used, as well as solving a number of linear systems of Euler's equations of the first order supporting your results.  . | |
| **Keywords:**  *Rung- kua method; approximate solution; Systems of Euler equations.*  Copyright©Authors, 2022, College of Sciences, University of Anbar. This is an open-access article under the CC BY 4.0 license ([http://creativecommons.org/licens es/by/4.0/](http://creativecommons.org/licens%20es/by/4.0/)). | |  |

**Introduction:**

The Euler equation is a model for the study of coefficient-variable equations used in modern economics and for calculating the cost of a given project. Some researchers have studied the Use Of Mathematical Systems in Cost Accounting

by Hayder N Kadhim, Athraa N Albukhuttar and Maha S ALibrahimi and others extended their search to Application of Linear Equation Systems in Banking Auditing By Hussein A ALMasoudi Some of them solve Euler's equations with classical methods and others with integral transformations .Differential equation systems exist in many scientific domains, including economics, computing and mechanics [1,5]. In economics, we're going to show how important Euler's formula is for auditing. The Euler equaion is he main link beween monetary policy and he real economy [8].

In addition, different approximate methods are used for solving these sysems of differenial equaions to obtain approximate solutions of mathematical problems [2,3].where the use of direct methods to solve these systems requires a significant computational effort, so many researchers turn to iterative methods (approximate methods) that do not calculate the direct solution, but begin with an approximate value, such as the Jacobi iterative method and the Gauss-Seidel method.

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**.** The researchers worked on developing these approaches, solving them more accurately and effectively, and applying them to a wide range of applications in other areas.

Rung -Kutta is one of the important and accurate numerical mehods for solving ordinary differential equations [4,6].This method depends on he initial values of the system, where he authors applied it in many mathematical applications[8,9,10].

In this work, mahemaical sysems of he Euler equaion of first order, they were solved with second - and fourth-order Runng-Kutta numerical methods, comparing the results of analytic solutions and the results of approximate solutions, and knowing the accuracy of the results, by calculating the relative error and our research was supported by some examples that show this.

**2-Preliminaries.**

In this section,we introduced some preliminaries which need in the following work.

**2.1. system of Euler's equations**.

An sysem of Euler's equation of first order has he form:

Ӽ

Assume

To convert initial condition

**Remarke 1.** When the exact solution for the system (4) the interval changes from the interval given in the system (1) because Euler equations ( where approaches 1 and t approaches 0.

Then from(2)and(3)convert to constant coefficients.

After solving the system (4) by classical methods whose solution with the variable t is by equation (2), we return the general formula to the variable Ӽ by and get the exact solution.

**2.2.Approximate Solutions for Systems of Euler's equations.**

In his section, Rung- Kua mehods applied for sysem (4) in dimension wo m=2.

**2.2.1. Second-order of Rung-kutta methods**

We generalize the method of Rung- Kutta :

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i=0,1,2, wih iniial condiion

**2.2.2. fourth-order of Rung-kutta methods**

We suffice with generalizing the traditional method of Rung-kutta.

(t,

(t,

And get the formula.

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i=0,1,2, wih iniial condiion

**Remark 2.** The error calculaed in hese problems is he relaive error symbolized .

**3-Application.**

In this section, some supported system as following.

**Example 1:-** To Solve the System.

Ӽ (ӽ)= Ⱬ (ӽ)(ӽ)

Ӽ (ӽ)=𝒴(ӽ)cos(ӽ) ,

With initial condition

(ӽ) = (1) = Ⱬ(ӽ) = Ⱬ(1) =

we get:

(t)= Ⱬ (t)(t)

(t)=(t)cos(t)

With initial conditions

(t) = (0) = Ⱬ(t) = Ⱬ(0) =

The exac solution for sysem(7) is

(t)=

Ⱬ(t)=

We solve his sysem of ordinary differenial equaion using Rung-Kua for second and fourh order on inervel [0,0.5] as in Table(3):

Table(1) The set solution of system(7)by second and fourth order of Runge-Kutta

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
|  | **Exact solution**  **𝒴()**  **Ⱬ()** | **RK**  **𝒴()**  **Ⱬ()** | **Error** | **RK**  **𝒴()**  **Ⱬ()** | **Error** |
| **0.1** | **0.04794255**  **0.08775825** | **0.04897127**  **0.08387912** | **2.145734843-E2**  **4.42024539-E2** | **0.04792963**  **0.08766190** | **2.695618556-E4**  **1.097902476-E4** |
| **0.2** | **0.16829419**  **0.10806046** | **0.16777382**  **0.11656720** | **3.092025934-E3**  **7.872204135-E2** | **0.16825748**  **0.10801803** | **2.181299307-E4**  **3.926505588-E4** |
| **0.3** | **0.29924849**  **0.02122116** | **0.21929155**  **0.022624593** | **2.671924594-E1**  **6.613366093-E2** | **0.29924787**  **0.02122638** | **2.071856737-E6**  **2.459808983-E4** |
| **0.4** | **0.36371897**  **-0.16645873** | **0.38780045**  **-0.16205711** | **6.620902946-E2**  **2.644271045-E2** | **0.36373284**  **-0.16647876** | **3.813384038-E5**  **1.203301263-E4** |
| **0.5** | **0.29923607**  **-0.40057180** | **0.29328164**  **-0.40679716** | **1.989877089-E2**  **1.554118388-E2** | **0.29925215**  **-0.40008644** | **5.373683727-E5**  **4.826004904-E3** |

After solve above system by using (1) we get:

𝒴( )=

Ⱬ( )=

We solve his sysemf ordinary differenial equaion using Rung-Kua for second and fourh order on inervel [1,1.5] as in Table(4):

Table(2) The set solution of system(5) by second and fourth order of Runge-Kutta

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
|  | **Exact solution**  **𝒴()**  **Ⱬ()** | **RK**  **𝒴()**  **Ⱬ()** | **Error** | **RK**  **𝒴()**  **Ⱬ()** | **Error** |
| **1.1** | **0.04792040**  **0.08779094** | **0.04897127**  **0.09387912** | **2.043451223-E2**  **6.934861388-E2** | **0.04792963**  **0.08766190** | **1.9261108-E4**  **1.46985554-E3** |
| **1.2** | **0.16821232**  **0.10806750** | **0.16777382**  **0.10656720** | **2.606824518-E3**  **1.013882989-E2** | **0.16815748**  **0.10891803** | **3.26016548-E4**  **7.80889995-E3** |
| **1.3** | **0.29921519**  **0.02128883** | **0.29929155**  **0.02624591** | **2.552009475-E3**  **2.32848399-E2** | **0.29924787**  **0.02128638** | **1.09219054-E4**  **1.15083825-E4** |
| **1.4** | **0.36378042**  **-0.16646065** | **0.36178004**  **-0.16205711** | **1.105070471-E2**  **2.645393971-E2** | **0.36333284**  **-0.16627876** | **1.23035765-E4**  **1.09269067-E4** |
| **1.5** | **0.29924059**  **-0.40054338** | **0.29328164**  **-0.40679716** | **1.991357523-E2**  **1.561324019-E2** | **0.29995215**  **-0.40008644** | **2.37788596-E4**  **1.14080028-E4** |

**Example2:-** To Solve the System .

Ӽ (ӽ)=3𝒴(ӽ)Ⱬ(ӽ)

Ӽ (ӽ)=4𝒴(ӽ)Ⱬ(ӽ) , (9)

With initial condition

(ӽ) = (1) = Ⱬ(ӽ) = Ⱬ(1) =

we get:

(t)=3(t)Ⱬ(t)

(t)=4(t)Ⱬ(t) ,

(t) = (0) =Ⱬ(t) = Ⱬ(0) =

The exac solution for sysem(11) is

(t)=

Ⱬ(t)=

We solvehis sysem of ordinary differenial equaion using Rung-Kua for second and fourh order on inervel [0,1] as in Table(1):

Table(3)The set solution of system(11) by second and fourth order of Runge-Kutta

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
|  | **Exact solution**  **𝒴()**  **Ⱬ()** | **RK**  **𝒴()**  **Ⱬ()** | **Error** | **RK**  **𝒴()**  **Ⱬ()** | **Error** |
| **0.2** | **0.21985249**  **0.56184526** | **0.22000000**  **0.56920000** | **6.709498719-E4**  **1.309033025-E2** | **0.21985333**  **0.56184666** | **3.82074362-E6**  **2.49178928-E6** |
| **0.4** | **0.23869196**  **0.62656637** | **0.23912000**  **0.62708000** | **1.793273641-E3**  **8.197535402-E4** | **0.23869413**  **0.62657005** | **9.09121530-E6**  **5.87328043-E6** |
| **0.6** | **0.25509663**  **0.69240514** | **0.25600000**  **0.69359000** | **3.541285512-E3**  **1.711223576-E3** | **0.25510087**  **0.69241238** | **1.66211525-E5**  **1.04563059-E5** |
| **0.8** | **0.26706491**  **0.75668391** | **0.26875000**  **0.75902000** | **6.309664568-E3**  **3.087273258-E3** | **0.26707221**  **0.75669651** | **2.73341788-E5**  **1.66516029-E5** |
| **1** | **0.27182818**  **0.81548454** | **0.27470000**  **0.81967000** | **1.056483548-E2**  **5.13248234-E3** | **0.27183995**  **0.81550501** | **4.32994106-E5**  **1.32021843-E4** |

After solve above system by using (1) we get:

𝒴()=

Ⱬ()=

We solvehis sysemf ordinary differenial equaion using Rung-Kua for second and fourh order on inervel [1,2] as in Table(2):

Table(4)The set solution of system(9)by second and fourth order of RungeKuttta

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
|  | **Exact solution**  **𝒴()**  **Ⱬ()** | **RK**  **𝒴()**  **Ⱬ()** | **Error** | **RK**  **𝒴()**  **Ⱬ()** | **Error** |
| **1.2** | **0.21982141**  **0.56184282** | **0.22000000**  **0.56920000** | **8.12432237-E3**  **1.30947299-E2** | **0.21985333**  **0.56184666** | **1.4520875-E4**  **6.8346517-E6** |
| **1.4** | **0.23869388**  **0.62658777** | **0.23912000**  **0.62708000** | **1.78521544-E3**  **7.85572307-E4** | **0.23869413**  **0.62657005** | **1.0473666-E6**  **2.8280156-E5** |
| **1.6** | **0.25509941**  **0.69249883** | **0.25600000**  **0.69359000** | **3.53034920-E3**  **1.57569941-E3** | **0.25510087**  **0.69241238** | **5.7232590-E6**  **1.2483775-E5** |
| **1.8** | **0.26709840**  **0.75669680** | **0.26875000**  **0.75902000** | **6.18348893-E3**  **3.07018610-E3** | **0.26707221**  **0.75669651** | **9.8053750-E5**  **3.8324464-E7** |
| **2** | **0.27182877**  **0.81544112** | **0.27470000**  **0.81967000** | **1.0562642-E2**  **5.1860028-E3** | **0.27183995**  **0.81550501** | **4.1129286-E5**  **7.8350230-E5** |

**Conclusions**

It was concluded that as the equation approached the polynomial, the error ratio was more similar to the wave equations. It was also concluded that the analytic solutions between the two systems converging to the results. Also, the analytic solutions of both systems with constant and variable coefficients were compared to the second - and fourth-order Runge-kutta method. It was found that the fourth order Runge-kutta had the lowest error ratio, making it more accurate in second-order results.

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**حل نظام معادلات أويلر باستخدام طرق رانج – كوتا**

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