Some Types of Lindelof in Bitopological Spaces

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ABSTRACT

In this paper, we define another types of Lindelof on bitopological space, namely N-Lindelof, S-Lindelof and pair-wise Lindelof spaces, and we introduce some properties about these types.

Introduction:

In 1963, the term of bitopological space was used for the first time by Kelly [1]. A set equipped with two topologies is called a bitopological space and denoted by \((X, \tau, \tau')\), where \((X, \tau)\) and \((X, \tau')\) are two topological spaces. In 2010 N.A.Jabbar and A.I.Nasir introduced N-open set [2]. A subset \(A\) of a bitopological space \((X, \tau, \tau')\) is called an N-open set if and only if it is open in the space \((X, \tau \vee \tau')\), where \(\tau \vee \tau'\) is the supremum topology on \(X\) contains \(\tau\) and \(\tau'\). In this paper, we introduce the concept of N-Lindelof space. A bitopological space \((X, \tau, \tau')\) is said to be an N-Lindelof space if and only if every N-open cover of \(X\) has a countable subcover, we study some properties of this kind of Lindelof space. Also we introduce the concept of S-Lindelof and pair-wise Lindelof. We also study the relationships among the three kinds of Lindelof spaces.

1. N-Lindelof Space

In this section, we give the definitions of N-open set, N-Lindelof space in bitopological spaces. open cover of \(X\) which is an N-Lindelof space. Therefore, there exists a countable number of \(\{U_\alpha : \alpha \in \Delta\} \ni \{X - A\} \cup \{U_\alpha : i \in \Delta \subset \mathbb{N}\}\) is a countable subcover of \(X\). Since \(A \subseteq X\) and \(X - A\) covers no part of \(A\), then \(\{U_\alpha : i \in \Delta \subset \mathbb{N}\}\) is a countable subcover of \(A\). So \(A\) is N-Lindelof set.

1.1 Definition [1]

Let \(X\) be a non-empty set, let \(\tau, \tau'\) be any two topologies on \(X\), then \((X, \tau, \tau')\) is called bitopological space.

1.2 Definition [2]

A subset \(A\) of a bitopological space \((X, \tau, \tau')\) is called an N-open set if and only if it is open in the space \((X, \tau \vee \tau')\), where \(\tau \vee \tau'\) is the supremum topology on \(X\) contains \(\tau\) and \(\tau'\).

1.3 Definition [2]

The complement of an N-open set in a bitopological space \((X, \tau, \tau')\) is called N-closed set.

1.4 Remark

Let \((X, \tau, \tau')\) be a bitopological space, then:

Every open set in \((X, \tau)\) or in \((X, \tau')\) is an N-open set in \((X, \tau, \tau')\).

Every closed set in \((X, \tau)\) or in \((X, \tau')\) is an N-closed set in \((X, \tau, \tau')\).

1.5 Note:

The opposite direction of remark 1.4 may be untrue as the following example shows:

1.11 Definition [2]

A function \(f : (X, \tau, \tau') \rightarrow (Y, T, T')\) is said to be an N-continuous function if and only if the inverse image of each N-open subset of \(Y\) is an N-open subset of \(X\).

1.12 Theorem

The N-continuous image of an N-Lindelof space is an N-Lindelof space.

Proof:

Let \((X, \tau, \tau')\) be an N-Lindelof space, and let
\( f : (X, \tau, \tau') \rightarrow (Y, T, T') \) be an \( N \)-continuous, onto function. To show that \( (Y, T, T') \) is an N-Lindelof space, let \( \{ U_\alpha : \alpha \in \Delta \} \) be an \( N \)-open cover of \( Y \), then \( \{ f^{-1}(U_\alpha) : \alpha \in \Delta \} \) is an \( N \)-open cover of \( X \), which is N-Lindelof space. So there exists a countable number of \( \{ f^{-1}(U_\alpha) : \alpha \in \Delta \} \) such that the family \( \{ f^{-1}(U_\alpha) : i \in \Delta \subset N \} \) covers \( X \) and since \( f \) is onto, then \( \{ U_{\alpha i} : i \in \Delta \subset N \} \) is a countable subcover of \( Y \). Hence \( Y \) is N-Lindelof space.

2. S-Lindelof Space

In this section, we give the definition of S-Lindelof space, then study pair-wise Lindelof space in bitopological spaces and relationships between them and the N-Lindelof.

2.1 Definition [3]

A subset \( A \) of a bitopological space \( (X, \tau, \tau') \) is said to be S-open set if it is \( \tau \)-open or \( \tau' \)-open. The complement of the S-open set is called S-closed set.

Example

Let \( X = \{1,2,3\} \), \( \tau = \{\phi, \{1\}, X \} \) and \( \tau' = \{\phi, \{2\}, X \} \)
then \( \tau \stackrel{\tau}{\cap} \tau' = \{\phi, \{1\}, \{2\}, \{1,2\}, X \} \) is the family of all N-open subsets of \( (X, \tau, \tau') \). \( \{1,2\} \) is an N-open set in \( (X, \tau, \tau') \) but it is not open in both \( (X, \tau) \) and \( (X, \tau') \). So \( \{3\} \) is an N-closed set in \( (X, \tau, \tau') \) which is not closed in both \( (X, \tau) \) and \( (X, \tau') \).

1.6 Definition [2]

Let \( (X, \tau, \tau') \) be a bitopological space, let \( A \) be a subset of \( X \). A subcollection of the family \( \tau \stackrel{\tau}{\cap} \tau' \) is called an N-open cover of \( A \) if the union of members of this collection contains \( A \).

1.7 Definition

A bitopological space \( (X, \tau, \tau') \) is said to be an N-Lindelof space if and only if every N-open cover of \( X \) has a countable subcover.

1.8 Corollary

If \( (X, \tau, \tau') \) is an N-Lindelof space. Then both \( (X, \tau) \) and \( (X, \tau') \) are Lindelof spaces.

Proof:

Follows from remark 1.4.

1.9 Corollary

If \( \tau \) is a subfamily of \( \tau' \). Then \( (X, \tau, \tau') \) is an N-Lindelof space if and only if \( (X, \tau) \) and \( (X, \tau') \) are Lindelof spaces.

Proof:

Necessity, follows from corollary (1.8). Sufficiency, in view of \( \tau \) is a subfamily of \( \tau' \), then \( \tau \cap \tau' = \tau' \). So \( (X, \tau, \tau') \) is N-Lindelof.

1.10 Corollary

The N-closed subset of an N-Lindelof space is N-Lindelof.

Proof:

Let \( (X, \tau, \tau') \) be an N-Lindelof space and let \( A \) be an N-closed subset of \( X \). To show that \( A \) is an N-Lindelof set. Let \( \{ U_\alpha : \alpha \in \Delta \} \) be an N-open cover of \( A \). Since \( A \) is N-closed subset of \( X \), then \( X \setminus A \) is an N-open subset of \( X \), so \( \{ X \setminus A \} \cup \{ U_\alpha : \alpha \in \Delta \} \) is an N-closed set. Hence there exists a countable number of \( \{ U_\alpha : \alpha \in \Delta \} \) \( \exists \) \( \{ X \setminus A \} \cup \{ U_{\alpha i} : i \in \Delta \subset N \} \) is a countable subcover of \( X \). Since \( A \subseteq X \) and \( X \setminus A \) covers no part of \( A \), then \( \{ U_{\alpha i} : i \in \Delta \subset N \} \) is a countable subcover of \( A \). So \( A \) is an S-Lindelof set.

2.8 Definition [4]

Let \( f : (X, \tau, \tau') \rightarrow (Y, T, T') \) be a function. Then \( f \) is said to be a bicontinuous function if and only if \( f^{-1}(U) \in \tau \) for each \( U \in T \), and \( f^{-1}(V) \in \tau' \) for each \( V \in T' \).

2.9 Example

Let \( X = \{1,2,3\} \), \( \tau = \{\phi, \{1\}, \{2\}, \{1,2\}\} \) and \( \tau' = \tau_D \). Let \( Y = \{a,b,c\} \) and \( T' = \{\phi, Y, \{a\}\} \). Define \( f : (X, \tau, \tau') \rightarrow (Y, T, T') \) \( \exists \) \( f(1) = a, f(2) = b \) and \( f(3) = c \). Then \( f \) is bicontinuous function. Where \( \tau_D \) and \( \tau_t \) are the discrete and indiscrete topologies on \( X \) and \( Y \) respectively.

2.10 Lemma

A bicontinuous image of an S-Lindelof space is an S-Lindelof space.

Proof:
Let $f : (X, \tau, \tau') \to (Y, T, T')$ be a bicontinuous, onto function and let $(X, \tau, \tau')$ be an S-Lindelof space. To show that $(Y, T, T')$ is an S-Lindelof, let \( \{ U_\alpha : \alpha \in \Delta \} \) be an S-open cover of \( Y \), then \( \{ f^{-1}(U_\alpha) : \alpha \in \Delta \} \) is an S-open cover of \( X \), which is an S-Lindelof space. Therefore there exists a countable number of \( \{ f^{-1}(U_\alpha) : \alpha \in \Delta \} \) such that the family \( \{ f^{-1}(U_\alpha) : i \in \Delta \subset N \} \) covers \( X \) and since \( f \) is onto, then \( \{ U_{\alpha_i} : i \in \Delta \subset N \} \) is a countable subcover of \( Y \). Hence \( Y \) is S-Lindelof space.

2.11 Corollary
Every N-Lindelof space is an S-Lindelof space. Proof:
Follows from remark 2.2.

2.2 Remark
Every S-open (S-closed) set in bitopological space \((X, \tau, \tau')\) is an N-open (N-closed) set.
The converse of the remark 2.2 need not be true, see the example of note (1.5), where the set \( \{ 1, 2 \} \) is N-open set which is not S-open. So \( \{ 3 \} \) is N-closed set which is not S-closed set.

2.3 Definition [3]
Let \((X, \tau, \tau')\) be a bitopological space, let \( A \) be a subset of \( X \). A subcollection of the family \( \tau \cup \tau' \) is called an S-open cover of \( A \) if the union of members of this collection contains \( A \).

2.4 Definition
A bitopological space \((X, \tau, \tau')\) is called an S-Lindelof if and only if every S-open cover of \( X \) has a countable subcover.

2.5 Corollary
If \((X, \tau, \tau')\) is an S-Lindelof space. Then both \((X, \tau)\) and \((X, \tau')\) are Lindelof spaces.

Proof:
Clear.

2.6 Corollary
If \( \tau \) is a subfamily of \( \tau' \). Then \((X, \tau, \tau')\) is an S-Lindelof space if and only if \((X, \tau)\) and \((X, \tau')\) are Lindelof spaces.

Proof:

2.7 Lemma
The S-closed subset of an S-Lindelof space is S-Lindelof.

Proof:
Let \((X, \tau, \tau')\) be an S-Lindelof space and let \( A \) be an S-closed subset of \( X \). To show that \( A \) is an S-Lindelof set. Let \( \{ U_\alpha : \alpha \in \Delta \} \) be an S-open cover of \( A \). Since \( A \) is S-closed subset of \( X \), then \( X - A \) is an S-open subset of \( X \), so \( \{ X - A \} \cup \{ U_\alpha : \alpha \in \Delta \} \) is an S-open cover of \( X \) which is an S-Lindelof space. Therefore, IF \( \tau \) is a subfamily of \( \tau' \) and \((X, \tau')\) is a Lindelof space, then \((X, \tau, \tau')\) is a pair-wise Lindelof space.

Proof:
Suppose that \( \tau \) is a subfamily of \( \tau' \) and \((X, \tau')\) be a Lindelof space. Then by corollary (1.9), \((X, \tau, \tau')\) is an N-Lindelof and by corollary (2.11), \((X, \tau, \tau')\) is S-Lindelof. Therefore \((X, \tau, \tau')\) is a pair-wise Lindelof.

2.19 Lemma
If \((X, \tau)\) and \((X, \tau')\) are Lindelof spaces. Then \((X, \tau, \tau')\) is S-Lindelof if and only if it is a pair-wise Lindelof.

Proof:
Necessities, follows from corollary (2.17).
Sufficiency, suppose \((X, \tau, \tau')\) is a pair-wise Lindelof space, to prove it is an S-Lindelof space, let \( \{ U_\alpha : \alpha \in \Delta \} \) be an S-open cover of \( X \). Then there are three probabilities.

IF \( \{ U_\alpha : \alpha \in \Delta \} \) is a \( \tau \)-open cover, since \((X, \tau)\) is Lindelof space, then \( \{ U_\alpha : \alpha \in \Delta \} \) has a countable subcover of \( X \), so \( X \) is an S-Lindelof.

IF \( \{ U_\alpha : \alpha \in \Delta \} \) is a \( \tau' \)-open cover, since \((X, \tau')\) is Lindelof space, then \( \{ U_\alpha : \alpha \in \Delta \} \) has a countable subcover of \( X \), so \( X \) is an S-Lindelof.

If \( \{ U_\alpha : \alpha \in \Delta \} \) is a pair-wise open cover, since \((X, \tau, \tau')\) is a pair-wise Lindelof space, then \( \{ U_\alpha : \alpha \in \Delta \} \) has a countable subcover, so \( X \) is an S-Lindelof.
From the above three probabilities we have $X$ is an S-Lindelof.

2.12 Corollary

Let $(X, \tau, \tau')$ be a bitopological space. If $\tau$ is a subfamily of $\tau'$, then the concepts of S-Lindelof and N-Lindelof are coincident.

Proof:

Follows from corollary (1.9) and corollary (2.6).

2.13 Definition [3]

Let $(X, \tau, \tau')$ be a bitopological space and let $A \subseteq X$. An S-open cover of $A$ is called a pair-wise open cover if it contains at least one non-empty element from $\tau$ and at least one non-empty element from $\tau'$.

2.14 Example

Let $X = \{1, 2, 3\}$, $\tau = \{\phi, X, \{1\}\}$. Then $\tau' = \{\phi, X, \{2\}, \{3\}, \{2, 3\}\}$. Then the cover $C = \{\{1\}, \{2\}, \{3\}\}$ is a pair-wise open cover of $X$.

2.15 Remark

It follows from the definition (2.13) that every pair-wise open cover of the bitopological space $(X, \tau, \tau')$ is an S-open cover.

The converse of the remark 2.15 need not be true, for example [2]:

Let $X = \{1, 2, 3\}$, $\tau = \{\phi, X, \{1\}\}$.

Let $\tau' = \{\phi, \{2\}, \{3\}, \{2, 3\}, \{1, 2\}, X\}$. Then the cover $C = \{\{1, 2\}, \{3\}\}$ is an S-open cover of $X$, but it is not pair-wise open cover.

2.16 Definition

A bitopological space $(X, \tau, \tau')$ is called a pair-wise Lindelof space if every pair-wise open cover of $X$ has a countable subcover.

2.17 Corollary

Every S-Lindelof space is a pair-wise Lindelof space.

Proof:

Follows from remark (2.15).

References


بعض أنواع فضاءات ليندلوف على الفضاءات الثنائية

فؤزي نوري نصار

الخلاصة:

في هذا البحث ، قمنا بتعريف أنواع أخرى من فضاءات ليندلوف على الفضاءات الثنائية أسماها فضاءات ليندلوف S وفضاءات ليندلوف N ، ودراسة بعض خواص هذه الفضاءات والعلاقة بينها.