Weakly Semi-2-Absorbing Submodules

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ABSTRACT:

In this paper we introduce and study the concept weakly semi-2-absorbing submodule as a generalization of 2-absorbing submodule, and give some of it is basic properties and characterization of this concept.


1. Introduction

Duns in 1980 introduce the concept of semi-prime submodule, where a submodule K of a module X is called semi-prime if \( b^2x \in K \), where \( b \in R \), \( x \in X \), it follows that \( bx \in K \) [1]. In [2], they introduced the concept of semi-2-absorbing as a generalization of semi-prime. This led us to introduce the concept weakly semi-2-absorbing submodule as a generalization of 2-absorbing submodule, where a submodule K of a module X is called 2-absorbing if ijx \( \in K \), with i, j \( \in R \), \( x \in X \), it follows that \( ixK \) or \( jxK \) or \( ijc[ K : X] = \{ r \in R : rX \subseteq K \} \) [3].

And a submodule K of X is called semi-2-absorbing if \( b^2x \in K \) with \( b \in R \), \( x \in X \), it follows that \( bx \in K \) or \( b^2 \in c[ K : X] \) [2].

In this work, all rings are commutative with identity and all modules are unitary R-modules.

2. Weakly semi-2-absorbing submodules

In this section, we introduce the concept of weakly semi-2-absorbing submodule, and give some basic properties of this concept.

2.1 Definition

A proper submodule K of X is called weakly semi-2-absorbing if \( 0 \neq b^2x \in K \) where \( b \in R \), \( x \in X \), it follows that bx \( \in K \) or \( b^2 \in c[ K : X] \). An ideal J of a ring R is called weakly semi-2-absorbing if \( b^2r \in J \) with \( b \), \( r \in R \), it follows that br \( \in J \) or \( b^2 \in J \).

2.2 Remark

Every 2-absorbing submodule is weakly semi-2-absorbing. However, the converse is not true.
Proof
Let K be a 2-absorbing submodule of a module X, and \(0 \neq b^2 x \in K\), where \(b \in R\), \(x \in X\). Then either \(bx \in K\) or \(b^2 \in [K : X]\), because K is 2-absorbing in X. It follows that K is a weakly semi-2-absorbing in X.

For the converse: consider \(X = Z \oplus Z\), \(R = Z\) and \(K = 10Z \oplus (0)\). It is clear that K is a weakly semi-2-absorbing, but not 2-absorbing in X.

2.3 PROPOSITION

Let \(E < X\), where X is R-module. If E is a weakly semi 2-absorbing in X. Then \([E : X]\) is weakly semi 2-absorbing ideal in R.

Proof
Let \(b, r \in R\), with \(0 \neq r \in [E : X]\). Then \((0) \neq b^2r x \in E\) for all nonzero \(x \in X\). Assume that \(b^2 \not\in [E : X]\), since E is a weakly semi-2-absorbing in X, then \(bx \in E\), it follows that \(br \in [E : X]\). Thus \([E : X]\) is a weakly semi-2-absorbing in R.

The converse of proposition (2.3) hold in the class of cyclic modules.

2.4 PROPOSITION

Let K be a proper submodule of cyclic module X, and \([K : X]\) is a weakly semi-2-absorbing ideal of R. Then K is a weakly semi-2-absorbing in X.

Proof
Assume that \(0 \neq b^2 x \in K\), where \(b \in R\), \(x \in X\), and \(X = R_m\), then there exist \(l \in R\) such that \(x = lm\). Thus \(0 \neq b^2 lm \in K\), it follows that \(0 \neq b^2 l \in [K : m] = [K : X]\), that is \(0 \neq b^2 l \in [K : X]\), it follows that \(bl \in [K : X]\) or \(0 \neq b^2 \in [K : X]\). That is \(blm \in K\) or \(b^2 \in [K : X]\), hence \(bx \in K\) or \(b^2 \in [K : X]\).

2.5 COROLLARY

Let K be a proper submodule of cyclic module X. Then K is a weakly semi-2-absorbing submodule iff \([K : X]\) is a weakly semi-2-absorbing ideal in R.

Proof
Direct

2.6 PROPOSITION

If a proper submodule K of a module X is a weakly semi-2-absorbing in X, then \([K : y]\) is a weakly semi-2-absorbing ideal in R for each nonzero \(y \in X - K\).

Proof
Assume that \(0 \neq b^2 \in [K : y]\), where \(b, l \in R\), then \(0 \neq b^2 ly \in K\), it follows that \(0 \neq b^2 (ly) \in K\), hence \(bly \in K\) or \(0 \neq b^2 le [K : X]\). Thus \(bl \in [K : y]\) or \(b^2 \epsilon [K : y]\) for each nonzero \(y \in X\). Thus \([K : y]\) is a weakly semi-2-absorbing in X.

2.7 PROPOSITION

A proper submodule K of a module X is a weakly semi-2-absorbing iff \([K : b^2 y] = [K : by]\) for all nonzero \(y \in \mathbb{X}\), \(0 \neq b \in R\) or \(b^2 \epsilon [K : X]\).

Proof

\((\rightarrow)\) let \(b^2 \not\in [K : X]\), to prove that \([K : b^2 y] = [K : by]\). We have \([K : by] \subseteq [K : b^2 y]\). now, let s be an nonzero element in \([K : b^2 y]\). Then \(0 \neq b^2 sy \in K\), it follows that \(s \epsilon K\) (since \(b^2 \not\in [K : X]\)), it follows that \(s \epsilon [K : by]\). Thus \([K : b^2 y] = [K : by]\).

\((\leftarrow)\) Assume that \([K : b^2 y] = [K : by]\) for each a nonzero \(y \in X\) and a nonzero \(b \in R\) or \(b^2 \epsilon [K : X]\), and \(0 \neq b^2 y \in K\), then we have \([K : b^2 y] = [K : by]\) or
b² ∈ [K : X]. If [K : b²y] = [K : by], and 0 ≠ b²y ∈ K, then we have [K : b²y] = R, so we have [K : by] = R, and hence b y ∈ K. Thus, K is a weakly semi-2-absorbing submodule in X.

2.8 REMARK
1. The intersection of two distinct weakly semi-2-absorbing submodules of a module X need not to be weakly semi-2-absorbing in X, as the following example: 4Z and 25Z are weakly semi-2-absorbing submodules of the Z-module Z. But 4Z ∩ 25Z = 100Z is not weakly semi-2-absorbing in Z, because 0 ≠ 5². 4 ∈ 100Z, but 5 . 4 = 20 ∈ 100Z and 5² = 25 ∈ [100Z : Z] = 100Z.

2. The intersection of two prime submodule of a module X is A weakly semi-2-absorbing module in X.

2.9 PROPOSITION
If K₁ is a weakly semi-2-absorbing submodule of a module X, then K₁ is a weakly semi-2-absorbing in K₂ where K₁ ⊆ K₂ is a submodule in X.

2.10 REMARK
A submodule of a weakly semi-2-absorbing module is not necessary weakly semi-2-absorbing. For example the submodules 36Z, 9Z of Z-module Z. 9Z is weakly semi-2-absorbing in Z and 36Z ⊆ 9Z is not weakly semi-2-absorbing in Z, because 0 ≠ 3². 4 ∈ 36Z, but 3 . 4 = 12 ∈ 36Z and 3² = 9 ∈ [36Z : Z] = 36Z.

2.11 PROPOSITION
Let K be a weakly semi-2-absorbing submodule of module X, with Kerf ⊆ K, where f : X → X be R-epimorphism. Then f(K) is a weakly semi-2-absorbing submodule of X.

Proof
Assume that 0 ≠ b²x ∈ f(K), where b ∈ R, x ∈ X then f(x) = x for some nonzero x in X. It follows that 0 ≠ b² f(x) ∈ f(K), then b² f(x) = b² f(k) for some nonzero k ∈ K. That is f(b²x - k) = 0, hence b²x - k ∈ Kerf ⊆ K, it follows that 0 ≠ b²x ∈ K, hence bx ∈ K or b² ∈ [K : X] because K is a weakly semi-2-absorbing. If bx ∈ K then f(bx) ∈ f(K), hence bx ∈ f(K).

If b² ∈ [K : X], then b² X ⊆ K, so that b² f(X) ⊆ f(K), it follows that b² X ⊆ f(K), then b² ∈ [f(X) : X].

2.12 COROLLARY
If K is a weakly semi-2-absorbing submodule of a module X, then K / H is a weakly semi-2-absorbing in X / H for some submodule H of X, with H ⊆ K.

Proof
Since \( \pi: X \to \frac{X}{H} \) defined by \( \pi(x) = x + H \) for all \( x \in X \) is an R-epimorphism with \( \text{Ker} f = H \). Hence the proof follows by proposition (2.12).

### 2.13 Proposition

The inverse image of a weakly semi-2-absorbing submodule is a weakly semi-2-absorbing.

**Proof**

Let \( f: X \to X \) be R-epimorphism and \( F \) a weakly semi-2-absorbing submodule of \( X \). Assume that \( 0 \neq b^2f^{-1}(F) \), where \( b \in R \times X \), then

\[ 0 \neq b^2f(y) \in F, \text{ it follows that } b^2x \subseteq F, \text{ because } F \text{ is weakly semi-2-absorbing in } X. \]

It follows that \( b \subseteq f^{-1}(F) \) or \( b \in \{ F \in X \} \).

If \( b \in \{ F \in X \} \), then \( b^2X \subseteq F \), then \( b^2(f(X)) \in F \), hence \( b^2X \subseteq f^{-1}(F) \). That is \( b^2 \in \{ f^{-1}(F) \} \).

### 2.14 Proposition

Let \( X = X_1 \oplus X_2 \) be a module, where \( X_1, X_2 \) are modules, and \( K_1 \) is a proper submodule of \( X_1 \). Then \( K_1 \) is a weakly semi-2-absorbing submodule of \( X_1 \) iff \( K_1 \oplus X_2 \) is a weakly semi-2-absorbing in \( X \).

**Proof**

\( \rightarrow \) Assume that \( 0 \neq b^2x \in K_1 \), where \( b \in R \) and \( x \) in \( X_1 \), then for any nonzero \( x \) in \( X_2 \), we have \( 0 \neq b^2(x_1, x_2) \in K_1 \oplus X_2 \). It follows that

\[ b(x_1, x_2) \in K_1 \oplus X_2 \text{ or } b^2 \in [K_1 \oplus X_2 : X_1 \oplus X_2] = [K_1 : X_1]. \]

Hence \( bx_1 \in K_1 \) or \( b^2 \in [K_1 : X_1] \).

### 2.15 Proposition

Let \( X = X_1 \oplus X_2 \) be a module, where \( X_1, X_2 \) are modules, and \( K_2 \) is a proper submodule of \( X_2 \). Then \( K_2 \) is a weakly semi-2-absorbing in \( X_2 \) iff \( X_1 \oplus K_2 \) is a weakly semi-2-absorbing in \( X \).

**Proof**

Similar as in proposition (2.14)

### 2.16 Proposition

Let \( Y \oplus Y \) be an R-module with \( \text{ann} Y + \text{ann} Y = R \), and \( T \) be a weakly semi-2-absorbing submodule of \( Y \oplus Y \). Then either

1. \( T = 0 \) and \( Y \) is a weakly semi-2-absorbing in \( Y \).
2. \( T = Y \oplus T \) and \( T \) is a weakly semi-2-absorbing in \( Y \).
3. \( T = Y \oplus T \) and \( T \) is a weakly semi-2-absorbing in \( Y \). Let \( 0 \neq x \in Y \), where \( b \in R, y \in Y \), then \( 0 \neq b^2x \in K_1 \) 

**Proof**

Since \( \text{ann} Y + \text{ann} Y = R \) and \( T \) is a submodule of \( Y \oplus Y \), then by [5, theo.2.4] \( T = T \oplus T \), hence we have (i) \( T \) is a submodule of \( Y \) and

\( T_2 = Y \oplus T \) (ii) \( T_1 = Y \) and \( T_2 \) is a submodule of \( Y \). From (i) we have \( T = T_1 \oplus Y \) or from (ii) we have \( T = Y \oplus T_2 \). Thus by proposition (2.15) we have \( T_1 \) is a weakly semi-2-absorbing in \( Y \) and \( T_2 \) is a weakly semi-2-absorbing in \( Y \). From (iii) we prove \( T_1 \) is a weakly semi-2-absorbing submodule in \( Y \), let \( 0 \neq b^2y \in T_1 \), where \( b \in R, y \in Y \), then \( 0 \neq b^2y \in T_1 \).
\[ \neq b^2(y, 0) \in T_1 \oplus T_2 = T. \] Thus \[ 0 \neq b^2(y, 0) \in T, \] but \( T \) is weakly semi-2-absorbing in \( Y \oplus Y' \), then either \( b(y,0) \in Y \) or \( b^2 \in [T : Y \oplus Y'] \subseteq [T_1 : Y] \), it follows that by \( \in T_1 \) or \( b^2 \in [T_1 : Y] \).

In the same way we can get \( T_2 \) is a weakly semi-2-absorbing in \( Y' \).

### 2.17 Proposition

Let \( Y \oplus Y' \) be an \( R \)-module, and \( T_1, T_2 \) are weakly semi-2-absorbing submodules in \( Y \) and \( Y' \) respectively, with \( [T_1 : Y] = [T_2 : Y] \). Then \( T = T_1 \oplus T_2 \) is a weakly semi-2-absorbing of \( Y \oplus Y' \).

#### Proof

Let \( 0 \neq b^2(y, y') \in T_1 \oplus T_2 \) for \( b \in R \), \( (y, y') \in Y \oplus Y' \), where \( y \) is a nonzero element in \( Y \) and \( y' \) is a nonzero element in \( Y' \). Then

\[ 0 \neq b^2y \in T_1 \] and \( 0 \neq b^2y' \in T_2. \] But \( T_1 \) and \( T_2 \) are weakly semi-2-absorbing in \( Y \) and \( Y' \) respectively, then by \( \in T_1 \) or \( b^2 \in [T_1 : Y] \) and \( \in T_2 \) or

\[ b^2 \in [T_2 : Y'] = [T_1 : Y], \] so by \( \in T_1 \) and \( \in T_2 \) or \( b^2 \in [T_1 : Y] \), thus

\[ b(y, y') \in T_1 \oplus T_2 \] or \( b^2 \in [T : Y \oplus Y'] \).

### 2.18 Proposition

Let \( Y \) be an \( R \)-module and \( T \) be a weakly semi-2-absorbing submodule of \( Y \). Then \( S^{-1}T \) is a weakly semi-2-absorbing submodule of \( S^{-1}Y \).

#### Proof

Let \( 0 \neq (x)^2y \in S^{-1}T, \) where \( x = \frac{x}{s_1} \in S^{-1}R, \) \( y = \frac{y}{s_2} \in S^{-1}Y \) and \( x \in R, s_1, s_2 \in S \).

Then \( 0 \neq (\frac{x}{s_1})^2 \cdot (\frac{y}{s_2}) \in S^{-1}T \), then \( \frac{x^2y}{s_1^2s_2} \in S^{-1}T \). That is \( \frac{x^2y}{t} \in S^{-1}T, \) where \( s_1^2s_2 = t \in S \).

Then there exist \( t \) in \( S \) with \( 0 \neq tx^2y \in T, \) it follows that \( x^2y \in T \) or \( x^2 \in [T : Y] \). Hence \( \frac{1}{s_1s_2t} \cdot \frac{x}{s_1} \frac{y}{s_2} \in S^{-1}T, \) or \( \frac{x}{s_1} \frac{y}{s_2} \in S^{-1}T : S^{-1}Y \).

Thus \( S^{-1}T \) is a weakly semi-2-absorbing in \( S^{-1}Y \).

### 2.19 Proposition

Every weakly semi-2-absorbing submodule of an \( R \)-module \( Y \) is weakly semi-2-absorbing in \( Y \).

#### Proof

Let \( T \) be a weakly semi-2-absorbing submodule of \( Y \). And \( 0 \neq b^2y \in T, b \in R, y \in Y \). That is \( 0 \neq b^2y \in T. \) Then either \( \in T \) or \( b^2 \in [T : Y] \).

### 2.20 Proposition

Every semi-2-absorbing submodule is weakly semi-2-absorbing

#### Proof

Clear

### 2.21 Proposition

Every semi-prime submodule is a weakly semi-2-absorbing

#### Proof

Since every prime submodule is a semi-prime \([6]\), we have the following corollary.

### 2.22 Corollary
Every prime submodule is a weakly semi-2-absorbing.

REFERENCES


