

N- Monotone Approximation in Weighted Space

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ABSTRACT

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The purpose of this paper is to discuss the degree of best monotone multi-approximation of unbounded functions weighted space in terms the modules of smoothness by using some algebraic linear operators. In addition, introduced ,proofs some properties of the modulus of smoothness.

Keywords:

Multi-unbounded functions.
 weighed space.
 monotone approximation.
 modulus of smoothness

1. INTRODUCTION

Many authors studied the problem of best monotone approximation for functions and operators in normed space also in metric space (cf.[1],[2],[3],[4],[5] and [6]).

The monotone approximation of periodic bounded functions by linear operator was obtained by ([7],[8],[9],[10],[11],[12],[13] and [14]).

Let $X = [-1,1]$, $L_{p,w}([-1,1])$ be the space , $1 \leq p < \infty$ of all unbounded functions with one variable and the norm given by

$$\|f\|_{p,w} = \left(\int_{-1}^1 (|f(t)w(t)|^p dt) \right)^{\frac{1}{p}} < \infty$$

Where $w(t) > 0$ is called weight function belong the set W of all weight functions .

For $k=1,2,.. . . .$ the k -modulus of smoothness of the function $f \in L_{p,w}([-1,1])$ is define by

$$\omega_k \left(f, \frac{1}{n} \right) = \text{Sub}_{|h| < \frac{1}{n}} \left\| \Delta_h^k f(\cdot) \right\|_{p,w}, n \in \mathbb{N}, \text{ where}$$

$$\Delta_h^k f(t) = \sum_{j=0}^k (-1)^{k+j} \binom{k}{j} f(x + jh), \text{ such that}$$

$\Delta_h^k f(t)$ is called k -th difference of f at point t with in quantity h . And the for $\Delta_h^k f(t)$ is define on real numbers denote by \mathbb{P}_n the set of algebraic polynomials of degree k and there $E_k(f, \frac{1}{n})_{p,w}$ be the best approximation of $f \in L_{p,w}([-1,1])$ by algebraic polynomials of \mathbb{P}_n

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. i.e

$$E_k \left(f, \frac{1}{n} \right)_{p,w} = \inf_{p_n \in \mathbb{P}_n} \{ \|f - p_n\|_{p,w} \}.$$

Let d is natural numbers and the space $L_{p,w}([-1,1]^d)$ of all unbounded functions of multi-variables , with $f \in L_{p,w}([-1,1]^d)$, given norm by

$$\|f\|_{p,w,([-1,1]^d)} = \left(\int_{([-1,1]^d)} |f(t)w(t)|^p dt \right)^{\frac{1}{p}} < \infty, 1 \leq p < \infty \text{ and } t \in ([-1,1]^d) .$$

2. Auxiliary lemmas

We will prove of the lemmas that we need our main results in next section.

Lemma 2.1 :

Let $f \in L_{p,w}([-1,1]^d)$, $1 \leq p < \infty$, $k \in \mathbb{N}$, $\delta > 0$. Then $\omega_k(f, \delta)_{p,w,([-1,1]^d)} \geq 0$.

Proof:

We have

$$\omega_k(f, \delta)_{p,w,([-1,1]^d)} = \sup_{|h| \leq \delta} \left\{ \left\| \Delta_h^k f(\cdot) \right\|_{p,w,([-1,1]^d)} \right\}$$

$$= \sup_{|h| \leq \delta} \left\{ \left\| \sum_{i=0}^k (-1)^{k-i} \binom{k}{i} f(\cdot + ih) \right\|_{p,w,([-1,1]^d)} \right\},$$

since

$$\left\| \sum_{i=0}^k (-1)^{k-i} \binom{k}{i} f(\cdot + ih) \right\|_{p,w,([-1,1]^d)} \geq 0,$$

$$\text{implies } \left\| \Delta_h^k f(\cdot) \right\|_{p,w,([-1,1]^d)} \geq 0.$$

$$\text{Hence } \sup_{|h| \leq \delta} \left\{ \left\| \Delta_h^k f(\cdot) \right\|_{p,w,([-1,1]^d)} \right\} \geq 0$$

$$\omega_k(f, \delta)_{p,w,([-1,1]^d)} \geq 0.$$

Lemma 2.2:

Let $f \in L_{p,w}([-1,1]^d)$, $1 \leq p < \infty$, $k \in \mathbb{N}$ and $\delta > 0$.
Then:

$$\omega_k(f, \delta)_{p,w,([-1,1]^d)} \rightarrow 0 \text{ as } \delta \rightarrow 0.$$

Proof:

$$\text{Let } \delta = \frac{1}{n}$$

$$\omega_k(f, \delta)_{p,w,([-1,1]^d)}$$

$$= \omega_k\left(f, \frac{1}{n}\right)_{p,w,([-1,1]^d)}$$

$$= \sup_{|h| \leq \frac{1}{n}} \left\{ \left\| \Delta_h^k f(\cdot) \right\|_{p,w,([-1,1]^d)} \right\}$$

$$= \sup_{|h| \leq \frac{1}{n}} \left\{ \left\| \Delta_h^{k-1} \Delta_h^1 f(\cdot) \right\|_{p,w,([-1,1]^d)} \right\}$$

$$= \sup_{|h| \leq \frac{1}{n}} \left\{ \left\| \Delta_h^{k-1} [f(\cdot) - \right.$$

$$\left. f(\cdot + \frac{1}{n}) \right\|_{p,w,([-1,1]^d)} \right\}.$$

If $n \rightarrow \infty$ then $\frac{1}{n} \rightarrow 0$

$$= \sup_{|h| \leq \frac{1}{n}} \left\{ \left\| \Delta_h^{k-1} [f(\cdot) - f(\cdot)] \right\|_{p,w,([-1,1]^d)} \right\}$$

$$= \sup_{|h| \leq \frac{1}{n}} \left\{ \left\| \Delta_h^{k-1} \cdot [0] \right\|_{p,w,([-1,1]^d)} \right\}$$

$$= \sup_{|h| \leq \frac{1}{n}} \|0\|_{p,w,([-1,1]^d)} = 0.$$

Lemma 2.3:

Let $f, g \in L_{p,w}([-1,1]^d)$, $1 \leq p < \infty$, $k \in \mathbb{N}$ and $\delta > 0$.

Then

$$\omega_k(f + g, \delta)_{p,w,([-1,1]^d)} \leq \omega_k(f, \delta)_{p,w,([-1,1]^d)} +$$

$$\omega_k(g, \delta)_{p,w,([-1,1]^d)}.$$

Proof:

$$\omega_k(f + g, \delta)_{p,w,([-1,1]^d)}$$

$$= \sup_{|h| \leq \delta} \left\{ \left\| \Delta_h^k (f + g)(\cdot) \right\|_{p,w,([-1,1]^d)} \right\}$$

$$= \sup_{|h| \leq \delta} \left\{ \left\| \Delta_h^k f(\cdot) + \Delta_h^k g(\cdot) \right\|_{p,w,([-1,1]^d)} \right\}$$

\leq

$$\sup_{|h| \leq \delta} \left\{ \left\| \Delta_h^k f(\cdot) \right\|_{p,w,([-1,1]^d)} + \left\| \Delta_h^k g(\cdot) \right\|_{p,w,([-1,1]^d)} \right\}$$

$$= \sup_{|h| \leq \delta} \left\{ \left\| \Delta_h^k f(\cdot) \right\|_{p,w,([-1,1]^d)} \right\} +$$

$$\sup_{|h| \leq \delta} \left\{ \left\| \Delta_h^k g(\cdot) \right\|_{p,w,([-1,1]^d)} \right\}$$

$$= \omega_k(f, \delta)_{p,w,([-1,1]^d)} + \omega_k(g, \delta)_{p,w,([-1,1]^d)}.$$

Lemma 2.4:

Let $f \in L_{p,w}([-1,1]^d)$, $1 \leq p < \infty$, $k \in \mathbb{N}$ and $\delta, c > 0$.

Then:

$$\omega_k(f, c\delta)_{p,w,([-1,1]^d)} \leq c^k \omega_k(f, \delta)_{p,w,([-1,1]^d)}.$$

Proof:

$$\omega_k(f, c\delta)_{p,w,([-1,1]^d)} = \sup_{|h| \leq c\delta} \left\{ \left\| \Delta_h^k f(\cdot) \right\|_{p,w,([-1,1]^d)} \right\}$$

$$\leq \sup_{|h| \leq c\delta} \left\{ \left\| \Delta_{\frac{h}{c}}^k f(\cdot) \right\|_{p,w,([-1,1]^d)} \right\}$$

$$= \sup_{|h| \leq c\delta} \left\{ \left\| (c\delta)^k D^k f(\cdot) \right\|_{p,w,([-1,1]^d)} \right\}$$

$$= c^k \sup_{|h| \leq c\delta} \left\{ \left\| \delta^k D^k f(\cdot) \right\|_{p,w,([-1,1]^d)} \right\}$$

$$= c^k \sup_{|h| \leq c\delta} \left\{ \left\| \Delta_{\delta}^k f(\cdot) \right\|_{p,w,([-1,1]^d)} \right\}$$

$$= c^k \omega_k(f, \delta)_{p,w,([-1,1]^d)}.$$

Lemma 2.5:

Let $f \in L_{p,w}([-1,1]^d)$, $1 \leq p < \infty$ and $k \in \mathbb{N}$. Then:

$$\omega_k(f, \delta_1)_{p,w,([-1,1]^d)} \leq \omega_k(f, \delta_2)_{p,w,([-1,1]^d)} \text{ for every } \delta_1 \leq \delta_2, \delta_1, \delta_2 > 0.$$

Proof:

$$\omega_k(f, \delta_1)_{p,w,([-1,1]^d)} = \sup_{|h| \leq \delta_1} \left\{ \left\| \Delta_h^k f(\cdot) \right\|_{p,w,([-1,1]^d)} \right\}$$

$$\leq \sup_{|h| \leq \delta_2} \left\{ \left\| \Delta_h^k f(\cdot) \right\|_{p,w,([-1,1]^d)} \right\} \text{ since } \delta_1 \leq \delta_2$$

$$= \omega_k(f, \delta_2)_{p,w,([-1,1]^d)}.$$

Lemma 2.6:

Let $f, \dot{f} \in L_{p,w}([-1,1]^d)$, $1 \leq p < \infty$, $k \in \mathbb{N}$ and $h > 0$.

Then

$$\omega_k(f, h)_{p,w,([-1,1]^d)} \leq \frac{h}{2} \omega_{k-1}(\dot{f}, h)_{p,w,([-1,1]^d)} \text{ where } \dot{f} \text{ is the first derivative of the function } f.$$

Proof:

$$\text{We have } \Delta_h^k f(x) = \Delta_h^{k-1}(\Delta_h^1 f(x))$$

$$= \Delta_h^{k-1} [f(x+h) - f(x-h)]$$

$$\left\| \Delta_h^k f(\cdot) \right\|_{p,w,([-1,1]^d)}$$

$$= \left\| \Delta_h^{k-1} [f(\cdot+h) \right.$$

$$\left. - f(\cdot-h)] \right\|_{p,w,([-1,1]^d)}$$

$=$

$$\left\| \Delta_h^{k-1} [f(\cdot+h) - f(\cdot) + f(\cdot) - f(\cdot-h)] \right\|_{p,w,([-1,1]^d)}$$

$$= \left\| \Delta_h^{k-1} [f(\cdot+h) - f(\cdot)] - \Delta_h^{k-1} [f(\cdot-h) - \right.$$

$$\left. f(\cdot)] \right\|_{p,w,([-1,1]^d)}$$

$$= \left\| \left[\Delta_h^{k-1} \int_0^{\frac{h}{4}} \dot{f}(\cdot+L) dL - \Delta_h^{k-1} \int_0^{\frac{h}{4}} \dot{f}(\cdot-L) dL \right] \right\|_{p,w,([-1,1]^d)}$$

\leq

$$\int_0^{\frac{h}{4}} \left\| \Delta_h^{k-1} [\dot{f}(\cdot+L) - \dot{f}(\cdot-L)] \right\|_{p,w,([-1,1]^d)} dL$$

$$\leq \int_0^{\frac{h}{4}} \omega_{k-1}(\dot{f}, \delta)_{p,w} dL \leq \frac{h}{2} \omega_{k-1}(\dot{f}, \delta)_{p,w,([-1,1]^d)}.$$

Lemma 2.7 : [15]

Let f be a bounded function that is measurable on the interval $[a, b]$ such that $a, b \in \mathbb{R}$. Then

$$\int_a^b f(x) dx \cong \sum_{i=1}^n c_i f(x_i), \text{ where } x_i = a + ih.$$

3. Main Results

In this section we prove our results by some types of linear operators to obtain the best monotone multi approximation of unbounded functions in weighted space.

Theorem 3.1 : Let $f \in L_{p,w}([-1,1]^d)$, $1 \leq p < \infty$ and define the operator

$$L_n(f, t) = (\Omega)^{\frac{1}{n}} \int_{[-1,1]^d} \dots \int_{[-1,1]^d} f(t + t_1 + \dots + t_k) dt , \quad \Omega = \frac{1}{k} \text{ Then}$$

- i - $L_n(f, t) \geq 0$.
- ii - $\|L_n(f, t)^k\|_{p,w,([-1,1]^d)} \leq C_{k1} \omega\left(f, \frac{1}{n}\right)_{p,w,([-1,1]^d)}$.
- iii - $\|f - L_n(f, \alpha)\|_{p,w,([-1,1]^d)} \leq C_{k2} \omega\left(f, \frac{1}{n}\right)_{p,w,([-1,1]^d)}$.
- iv - $\|f - L_n(f, \alpha)\|_{p,w,([-1,1]^d)} \leq C_{k3} \omega\left(f', \frac{1}{n}\right)_{p,w,([-1,1]^d)}$.

Proof :

$$i - L_n(f, t) = (\Omega)^{\frac{1}{n}} \int_{[-1,1]^d} \dots \int_{[-1,1]^d} f(t + t_1 + \dots + t_k) dt$$

$$\int_{[-1,1]^d} \dots \int_{[-1,1]^d} f(t + t_1 + \dots + t_k) dt \geq 0$$

Thus, $L_n(f, t) \geq 0$

$$ii - L_n(f, t) = (\Omega)^{\frac{1}{n}} \int_X \dots \int_X f(t + t_1 + \dots + t_k) dt$$

$$\|L_n(f, t)\|_{p,w,([-1,1]^d)} \leq \|L_n(f, t) - g(t) + g(t)\|_{p,w,([-1,1]^d)}$$

$$\leq \|L_n(f, t) - g(t)\|_{p,w,([-1,1]^d)} + \|g(t)\|_{p,w,([-1,1]^d)}$$

$$\leq \text{Max}(c) \|\Delta_{\delta}^1 f(t)\|_{p,w,([-1,1]^d)}$$

$$\leq \text{Max}(c) \text{Sup} \|\Delta_{\delta}^1 f(t)\|_{p,w,([-1,1]^d)}$$

$$\leq \text{Max}(C) \omega\left(f, \frac{1}{n}\right)_{p,w,([-1,1]^d)}$$

$$iii - \|f - L_n(f, \alpha)\|_{p,w,([-1,1]^d)}$$

$$= \left(\int_X |f(x) - L_n(f) w(x)|^p dx\right)^{\frac{1}{p}}$$

$$\leq \text{Sup} \left(\int_X |f(x) - L_n(f) w(x)|^p dx\right)^{\frac{1}{p}} =$$

$$\leq C_{k2} \text{Sup} (\|\Delta_{\delta}^1 f(\cdot)\|_{p,w,([-1,1]^d)})$$

$$\leq C_{k2} \omega\left(f, \frac{1}{n}\right)_{p,w,([-1,1]^d)}$$

iv- from (iii) by clear

$$\|f - L_n(f)\|_{p,w,([-1,1]^d)} \leq C_{k2} \omega\left(f, \frac{1}{n}\right)_{p,w,([-1,1]^d)}$$

from lemma (2.6)

$$\|f - L_n(f)\|_{p,w,([-1,1]^d)} \leq C_{k3} \omega\left(f', \frac{1}{n}\right)_{p,w,([-1,1]^d)}$$

Theorem 3.2 : Let $f \in L_{p,w}([-1,1]^d)$, $1 \leq p < \infty$ and there exist class of algebraic Polynomials $\{G_k\}_{k=1}^n$ satisfies

$$\int_x G_k(x) dx = 1$$

$$\|G_k\|_{p,w} \leq C_k , \quad \text{Where } C_k \text{ is positive constant ,}$$

if f satisfies Theorem 3.1

$\lambda = f_n(f, \delta)$, $\delta = \frac{1}{n}$ and $Cf_n = \int_{-\frac{1}{2}}^{\frac{1}{2}} \lambda(x) G_k(y - x) dx$. Then

$$1- \|f - f^*\|_{p,w,([-1,1]^d)} \leq C_k \omega_k(f, \delta)_{p,w}$$

$$1- \|f - f^*\|_{p,w,([-1,1]^d)} \leq C_k \omega_k(f', \delta)_{p,w}$$

Proof :

$$\text{Let } G_k = G_{k+1} = \dots = G_{4k-1} = \frac{1}{2} , \quad n \geq 2k ,$$

$$\text{Let } G_{4n-4k}(x) = C_n \left[(p_{2n}(x) / x^2 - t_{1,2n}^2) \dots (x^2 - t_{k,2n}^2) \right]^2 \dots \dots (1)$$

Where P_{2n} is algebraic Polynomial of degree $2n$ and $t_{1,2n}, \dots, t_{n,2n}$ are its positive Zeros in creasing order. Define the remaining G_n with the relation

$$G_{4n+1} = G_{4n+2} = G_{4n+3} = G_{4n} , \quad n \geq k$$

We have

$$C_1 n^{-1} \leq t_{1,2n} \leq \dots \leq t_{k,2n} \leq C_2 n^{-1} , \quad n > k \dots \dots (2)$$

By using $\|P_n\|_{p,w} = 1$

Let last equality being a consequence of starlings formula (1)and(2) to gather imply

$$G_{4n-4k}(0) \geq C c_n n^{4k-1} , \quad n \geq 2k \dots \dots \dots (3)$$

Write

$$1 = \int_{-1}^1 G_{4n-4k}(x) dx = \sum_{k=-n}^n B_k (2n+1) G_{4n-4k}(t_{k,2n+1})$$

Where $B_k (2n+1)$ are the weights of Gaussian quadrature formula , exact for Polynomials of degree $4n+1$, with nodes The zeros of the Legendre of degree $2n+1$ Therefore

$$1 \geq B_0 (2n+1) G_{4n-4k}(0)$$

$$\text{Since } B_0 (2n+1) = \pi(1+0(1)) / (2n+1)$$

Then

$$G_{4n-4k}(0) \leq C . n \dots \dots \dots (4)$$

From (3) and (4) imply $C_n \leq C_n^{2-4k}$

By using The definition of the G_n and (2) implies

$$\|G_n\|_{p,w,([-1,1]^d)} \leq C_n^{2-4k}$$

Again let $n \geq 2k$ $X^{2k} G_{4n-4k}(x)$ is polynomial of degree $4n - 2k$

Therefore for $i = 1, 2, \dots, k$

$$\begin{aligned} Hi &= \int_{-1}^1 a^{2i} G_{4n-4k}(x) dx \\ &= 2 \sum b_{j,2n}^{2i} B_j(2n) G_{4n-4k}(b_{j,2n}) \end{aligned}$$

Where $B_j(2n)$ are the weights of the Gaussian quadrature formula, exact for polynomials of degree $4n-1$, with nodes The zeros of the Legendre polynomial of degree $2n$.

Since G_{4n-4k} has zeros at $t_{k+1, 2n}, \dots, t_{n, 2n}$

$$\text{Then } Hi = 2 \sum_{j=1}^k t_{j,2n}^{2i} B_j(2n) G_{4n-4k}(t_{j,2n})$$

Since G_{4n-4k} has a local maximum on $[-t_{k+1, 2n}, t_{k+1, 2n}]$

At zero. Then

$$B_j(2n) \leq \frac{\pi}{2n} (1 + o(1)), j = 1, 2, \dots, k$$

From (2) and (4)

The definition of the G_{n1} imply

$$\int_{-1}^1 x^{2i} G_n(x) dx \leq C_n^{-2i}, \quad i = 1, 2, \dots, n \geq k \dots (5)$$

By using (5) we get

$$\begin{aligned} 1 - F_n(1, t) &= \int_{-1}^1 G_n(x) dx - \int_{-\frac{1}{2}}^{\frac{1}{2}} G_n(x) dx \leq \\ &2 \int_{\frac{1}{4}}^1 G_n(x) dx \leq C_n^{2-4k} \\ F_n((x, t)^{2i}, t) &= \int_{-\frac{1}{2}}^{\frac{1}{2}} (x-t)^{2i} G_n(x-t) dx \\ &= \int_{-\frac{1}{2}}^{\frac{1}{2}} x^{2i} G_n(x) dx \\ &\leq \int_{-1}^1 (x)^{2i} G_n(x) dx \end{aligned}$$

And applying (5),

$$F_n((x-t)^{2i}, t) \leq C^{-2n}, \quad i = 1, 2, \dots, k \dots (6)$$

$$F_n(|x-t|^k, t) \leq \int_{-1}^1 |x|^k G_n(x) dx$$

$$\leq \left[\int_{-1}^1 x^{2k} G_n(x) dx \right]^{\frac{1}{2}} \leq C_n^{-k}$$

By using The Schwartz inequality and (6) for i is odd, Then

$$\begin{aligned} |F_n((x-t)^i, t)| &= \left| \int_{-\frac{1}{2}}^{\frac{1}{2}} x^i G_n(x) dx \right| \\ &\leq 2 \int_{-\frac{1}{4}}^{\frac{1}{4}} x^i G_n(x) dx \end{aligned}$$

Since G_n is even, Then

$$|F_n((x-t)^i, x)| \leq C_n^{2-4k}, i = 1, 3, 5 \dots$$

If $f \in [-\frac{1}{2}, \frac{1}{2}]$, Then Taylor's theorem gives

$$\begin{aligned} \lambda(x) &= \left[\sum_{i=0}^{k-1} \frac{\lambda^{(i)}(t)(x-t)^i}{i!} \right] \\ &+ \frac{1}{(k-1)!} \int_t^x \lambda^{(k)}(u) (x-u)^{k-1} du \end{aligned}$$

Since the last term on the right hand side is bounded in modulus by

$$\left(\frac{1}{k!}\right) |x-t|^k \cdot \|\lambda^{(k)}\|_{p,w}$$

$$\begin{aligned} |F_n(\lambda, t) - \lambda(t)| &\leq |\lambda(t)| |\lambda - F_n(\lambda)| \\ &+ \sum_{i=1}^{k-1} \frac{|\lambda^{(i)}(t)|}{i!} \cdot |F_n(x-t)^i, t| + \end{aligned}$$

$$\left(\frac{1}{k!}\right) \cdot \|\lambda^{(k)}\|_{p,w} F_n(|u|^k, f(u))$$

Thus $\|F_n(\lambda, t) - \lambda(t)\|_{p,w} \leq C \cdot \|\lambda\|_{p,w,([-1,1]^d)} +$

$$\sum_{i=1}^{k-1} \frac{\|\lambda^{(i)}\|_{p,w}}{i!} \cdot \|F_n(u)^i, f(u)\|_{p,w,([-1,1]^d)} +$$

$$\left(\frac{1}{k!}\right) \|\lambda^{(k)}\|_{p,w} \cdot \|F_n((u)^k, f(u))\|_{p,w,([-1,1]^d)}$$

$\leq \text{Max}(c) \cdot w_k(f, \delta)_{p,w,([-1,1]^d)}$.

We have, from Theorem 3.1

$\|f\| \leq C \|f'\|$ implies

$$\|f - f^*\| \leq \text{Max}(c) \omega_k(f', \delta)_\alpha$$

Theorem 3.3: Let $\in L_{p,w}([-1,1]^d)$, $1 \leq p < \infty$ and $r \in \mathbb{N}$. Then

$$E_r(f, \delta)_{p,w} \leq \text{Max}(c) \omega_r(f, \delta)_{p,w}$$

Proof:- Let $f = \alpha - \beta$ such that

$$\alpha(x) = \beta(-1) + 2(\beta(1) - \beta(-1))(x + 1)$$

From of property of smoothness of modulus , we have

$$\omega_r(f, \delta)_{p,w,([-1,1]^d)} \leq C\omega_r(\alpha, \delta)_{p,w,([-1,1]^d)}$$

Theorem 3.1 and 3.2 apply to , writing

$$\bar{x}_k(\alpha) = \beta(x) + x_n^* (x_k(f))$$

Theorem 3.1 and 3.2 imply

$$\begin{aligned} \|f - \bar{x}_n(f)\|_{p,w,([-1,1]^d)} &= \left(\int_X |\alpha(x) - x(f, x).w(x)|^p dx \right)^{\frac{1}{p}} \\ &\leq \left(\int_{X^d} |\alpha(x) - g(x).w(x)|^p dx \right)^{\frac{1}{p}} \\ &\quad + \left(\int_{X^d} |g(x) - x(f, x).w(x)|^p dx \right)^{\frac{1}{p}} \end{aligned}$$

Let $g = x(f)$ Then

$$\bar{x}_n(f) = \beta(x) + x^*(g) = \beta(x) + \int_{X^d} g(t). \lambda_n(t - x) dt$$

$$\bar{x}_n(\alpha, x)' = \beta'(x) + \int_X g(t). -\lambda'_n(t - x) dt$$

$$= \beta'(x) + [-g(t)\lambda_n(t - x)]_{X^d} + \int_{X^d} g(t)\lambda_n(t - x) dt$$

$r \geq 2$ alternate differen and integration by parts yield :

$$\begin{aligned} \bar{x}_k(\alpha, x)^r &= (-1)^n \left[\sum_{j=0}^{r-1} (-1)^j g^j(t) \lambda_n(r - 1 - j)(t - x) \right]_{X^d}^{\frac{1}{2}} \\ &\quad + \int_{X^d} g^r(t) \lambda_n(t - x) dt \\ &= \gamma(x) + \int_{X^d} g^r(t) \lambda_n(t - x) dt \end{aligned}$$

From Theorem 3.1 , we get

$$\|f(\alpha) - r\|_{p,w,([-1,1]^d)} \leq \|\Delta_{\alpha}^r f\|_{p,w,([-1,1]^d)}$$

$$\|f(\alpha) - r\|_{p,w,([-1,1]^d)} \leq \sup \|\Delta_{\alpha}^r f\|_{p,w,([-1,1]^d)}$$

$$E_r(f, \delta)_{p,w,([-1,1]^d)} \leq \text{Max}(c) \omega_r(f, \delta)_{p,w,([-1,1]^d)} .$$

4. Conclusion

In this work, we managed of prove some properties modulus of smoothness which need it in main results proofs and we can found the degree of best monotone multi-approximation of unbounded functions by using some types of linear operators and algebraic polynomials.

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أفضل تقريب متعدد للدوال الغير مقيدة بواسطة مقياس النعومة

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الخلاصة:

الغرض من هذا البحث هو مناقشة درجة افضل تقريب افضل ترتيب متعدد للدوال الغير مقيدة في الفضاء الموزون بشروط مقياس النعومة بواسطة بعض المؤثرات الخطية الجبرية وكذلك عرض براهين بعض خواص مقياس النعومة .