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N- Monotone Approximation in Weighted Space

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ABSTRACT

The purpose of this paper is to discuss the degree of best monotone multi-approximation of unbounded functions weighted space in terms the modules of smoothness by using some algebraic linear operators. In addition, introduced ,proofs some properties of the modulus of smoothness.

1. INTRODUCTION

Many authors studied the problem of best monotone approximation for functions and operators in normed space also in metric space (cf.[1] ,[2],[3],[4],[5] and [6]).

The monotone approximation of periodic bounded functions by linear operatory was obtained by ([7],[8],[9],[10],[11],[12],[13] and [14]) .

Let X= $[-1,1]$, $L_{p,w}([-1,1])$ be the space , $1 \le p < \infty$ of all unbounded functions with one variable and the norm given by

$$\|f\|_{p,w} = (\int_{1}^{-1} (|f(t)w(t)|^p dt)^{\frac{1}{p}} < \infty$$

Where $w(t) > 0$ is calledweight function belong the set W of all weight functions .

For k=1,2,. . . . . . the k-modulus of smoothness of the function $f \in L_{p,w}([-1,1])$ is define by

$\omega_k\left(f,\frac{1}{n}\right) = Sub_{|h|<\frac{1}{n}}\left\|\Delta_h^k f(.)\right\|_{p,w}, n \in \mathbb{N}$, where

$\Delta_h^k f(t) = \sum_{j=0}^{k}(-1)^{k+j} \binom{k}{j} f(x+jh)$, such that $\Delta_h^k f(t)$ is called k-th difference of f at point t with in quantity h . And the for $\Delta_h^k f(t)$ is define on real numbers denote by $\mathbb{P}_n$ the set of algebraic polynomials of degree k and there $E_k(f,\frac{1}{n})_{p,w}$ be the best approximation of $f \in L_{p,w}([-1,1])$ by algebraic polynomials of $\mathbb{P}_n$. i.e

$$E_k\left(f,\frac{1}{n}\right)_{p,w} = inf_{p_n \in \mathbb{P}_n}\{\|f-p_n\|_{p,w}\}.$$

Let d is natural numbers and the space $L_{p,w}([-1,1]^d)$ of all unbounded functions of malti-variables , with $f \in L_{p,w}([-1,1])^d$, given norm by

$\|f\|_{p,w,([-1,1])^d} = \left(\int_{([-1,1])^d}|f(t)w(t)|^p dt\right)^{\frac{1}{p}} < \infty,\ 1 \le p < \infty$ and
$t \in ([-1,1])^d$.

2. Auxiliary lemmas

We will prove of the lemmas that we need our main results in next section.

Lemma 2.1 :
Let$f \in L_{p,w}([-1,1]^d),\ 1 \le p < \infty,\ k \in \mathbb{N},\ \delta > 0$. Then

$$\omega_k(f,\delta)_{p,w,([-1,1])^d} \ge 0.$$

Proof:
We have

$$\omega_k(f,\delta)_{p,w,([-1,1]^d)} = \sup_{|h|\le\delta}\left\{\left\|\Delta_h^k f(.)\right\|_{p,w([-1,1])^d,}\right\}$$

$$= \sup_{|h|\le\delta}\left\{\left\|\sum_{i=0}^{k}(-1)^{k-i}\binom{k}{i}f(.+ih)\right\|_{p,w,([-1,1]^d)}\right\},$$

since

$$\left\|\sum_{i=0}^{k}(-1)^{k-i}\binom{k}{i}f(.+ih)\right\|_{p,w([-1,1]^d)} \ge 0,$$

implies $\left\|\Delta_h^k f(.)\right\|_{p,w([-1,1]^d)} \ge 0.$

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Hence $\sup_{|h|\le\delta}\left\{\left\|\Delta_h^k f(.)\right\|_{p,w,([-1,1]^d)}\right\} \ge 0$

$$\omega_k(f,\delta)_{p,w,([-1,1]^d)} \ge 0.$$

Lemma 2.2:
Let Let$f \in L_{p,w}([-1,1]^d)$, $1 \le p < \infty$, $k \in \mathbb{N}$ *and* $\delta > 0$.
Then:

$$\omega_k(f,\delta)_{p,w,([-1,1]^d)} \to 0 \quad as \quad \delta \to 0.$$

Proof:
Let $\delta = \frac{1}{n}$

$$\omega_k(f,\delta)_{p,w,([-1,1]^d)} = \omega_k\left(f,\frac{1}{n}\right)_{p,w,([-1,1]^d)} = \sup_{|h|\le\frac{1}{n}}\left\{\left\|\Delta_h^k f(.)\right\|_{p,w,([-1,1]^d)}\right\}$$

$$= \sup_{|h|\le\frac{1}{n}}\left\{\left\|\Delta_h^{k-1}\Delta_h^1 f(.)\right\|_{p,w,([-1,1]^d)}\right\} = \sup_{|h|\le\frac{1}{n}}\left\{\left\|\Delta_h^{k-1}\left[f(.) - f(.+\tfrac{1}{n})\right]\right\|_{p,w,([-1,1]^d)}\right\}.$$

If $n \to \infty$ then $\frac{1}{n} \to 0$

$$= \sup_{|h|\le\frac{1}{n}}\left\{\left\|\Delta_h^{k-1}[f(.) - f(.)]\right\|_{p,w,([-1,1]^d)}\right\}$$

$$= \sup_{|h|\le\frac{1}{n}}\left\{\left\|\Delta_h^{k-1}.[0]\right\|_{p,w,([-1,1]^d)}\right\}$$

$$= \sup_{|h|\le\frac{1}{n}}\|0\|_{p,w,([-1,1]^d)} = 0.$$

Lemma2.3:
Let $f, g \in L_{p,w}([-1,1]^d)$, $1 \le p < \infty$, $k \in \mathbb{N}$ and $\delta > 0$.
Then

$$\omega_k(f+g,\delta)_{p,w,([-1,1]^d)} \le \omega_k(f,\delta)_{p,w,([-1,1]^d)} + \omega_k(g,\delta)_{p,w,([-1,1]^d)}.$$

Proof:

$$\omega_k(f+g,\delta)_{p,w,([-1,1]^d)}$$

$$= \sup_{|h|\le\delta}\left\{\left\|\Delta_h^k (f+g)(.)\right\|_{p,w,([-1,1]^d)}\right\}$$

$$= \sup_{|h|\le\delta}\left\{\left\|\Delta_h^k f(.) + \Delta_h^k g(.)\right\|_{p,w,([-1,1]^d)}\right\}$$

$$\le \sup_{|h|\le\delta}\left\{\left\|\Delta_h^k f(.)\right\|_{p,w,([-1,1]^d)} + \left\|\Delta_h^k g(.)\right\|_{p,w,([-1,1]^d)}\right\}$$

$$= \sup_{|h|\le\delta}\left\{\left\|\Delta_h^k f(.)\right\|_{p,w,([-1,1]^d)}\right\} + \sup_{|h|\le\delta}\left\{\left\|\Delta_h^k g(.)\right\|_{p,w,([-1,1]^d)}\right\}$$

$$= \omega_k(f,\delta)_{p,w,([-1,1]^d)} + \omega_k(g,\delta)_{p,w,([-1,1]^d)}.$$

Lemma 2.4:
Let $f \in L_{p,w}([-1,1]^d)$, $1 \le p < \infty$, $k \in \mathbb{N}$ and $\delta, c > 0$.
Then:

$$\omega_k(f,c\delta)_{p,w,([-1,1]^d)} \le c^k\omega_k(f,\delta)_{p,w,([-1,1]^d)}.$$

Proof:

$$\omega_k(f,c\delta)_{p,w,([-1,1]^d)} = \sup_{|h|\le c\delta}\left\{\left\|\Delta_h^k f(.)\right\|_{p,w,([-1,1]^d)}\right\}$$

$$\le \sup_{|h|\le c\delta}\left\{\left\|\Delta_{c\delta}^k f(.)\right\|_{p,w,([-1,1]^d)}\right\}$$

$$= \sup_{|h|\le c\delta}\left\{\left\|(c\delta)^k D^k f(.)\right\|_{p,w,([-1,1]^d)}\right\}$$

$$= c^k \sup_{|h|\le c\delta}\left\{\left\|\delta^k D^k f(.)\right\|_{p,w,([-1,1]^d)}\right\}$$

$$= c^k \sup_{|h|\le c\delta}\left\{\left\|\Delta_\delta^k f(.)\right\|_{p,w,([-1,1]^d)}\right\}$$

$$= c^k\omega_k(f,\delta)_{p,w,([-1,1]^d)}.$$

Lemma 2.5:
Let $f \in L_{p,w}([-1,1]^d)$, $1 \le p < \infty$ and $k \in \mathbb{N}$. Then:
$\omega_k(f,\delta_1)_{p,w,([-1,1]^d)} \le \omega_k(f,\delta_2)_{p,w,([-1,1]^d)}$ for every $\delta_1 \le \delta_2$, $\delta_1, \delta_2 > 0$.

Proof:

$$\omega_k(f,\delta_1)_{p,w,([-1,1]^d)} = \sup_{|h|\le\delta_1}\left\{\left\|\Delta_h^k f(.)\right\|_{p,w,([-1,1]^d)}\right\}$$

$$\le \sup_{|h|\le\delta_2}\left\{\left\|\Delta_h^k f(.)\right\|_{p,w,([-1,1]^d)}\right\} \text{ since } \delta_1 \le \delta_2$$

$$= \omega_k(f,\delta_2)_{p,w,([-1,1]^d)}.$$

Lemma 2.6:
Let $f, \acute{f} \in L_{p,w}([-1,1]^d)$, $1 \le p < \infty$, $k \in \mathbb{N}$ and $h > 0$.
Then
$\omega_k(f,h)_{pp,w,([-1,1]^d)} \le \frac{h}{2}\,\omega_{k-1}(\acute{f},h)_{p,w,([-1,1]^d)}$ where $\acute{f}$ is the first derivative of the function f.

Proof:
We have $\Delta_h^k f(x) = \Delta_h^{k-1}(\Delta_h^1 f(x))$

$$= \Delta_h^{k-1}[f(x+h) - f(x-h)]$$

$$\left\|\Delta_h^k f(.)\right\|_{p,w,([-1,1]^d)} = \left\|\Delta_h^{k-1}[f(.+h) - f(.-h)]\right\|_{p,w,([-1,1]^d)}$$

$$= \left\|\Delta_h^{k-1}[f(.+h) - f(.) + f(.) - f(.-h)]\right\|_{p,w,([-1,1]^d)}$$

$$= \left\|\Delta_h^{k-1}[f(.+h) - f(.)] - \Delta_h^{k-1}[f(.-h) - f(.)]\right\|_{p,w,([-1,1]^d)}$$

$$= \left\|\Delta_h^{k-1}\int_0^{\frac{h}{4}} \acute{f}(.+L)dL - \Delta_h^{k-1}\int_0^{\frac{h}{4}} \acute{f}(.-L)dL\right\|_{p,w,([-1,1]^d)}$$

$$\le \int_0^{\frac{h}{4}} \left\|\Delta_h^{k-1}[\acute{f}(.+L) - \acute{f}(.-L)]\right\|_{p,w,([-1,1]^d)} dL$$

$$\le \int_0^{\frac{h}{4}} \omega_{k-1}(\acute{f},\delta)_{p,w}\, dL \le \frac{h}{2}\omega_{k-1}(\acute{f},\delta)_{p,w,([-1,1]^d)}.$$

Lemma 2.7 : [15]
Let f be a bounded function that is measurable on the interval [a,b] such that $a, b \in \mathbb{R}$. Then
$\int_a^b f(x)dx \cong \sum_{i=1}^{n} c_i\, f(x_i)$, where $x_i = a + ih$.

3. Main Results

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In this section we prove our results by some types of linear operators to obtain the best monotone multi approximation of unbounded functions in weighted space.

Theorem 3.1 : Let $f \in L_{p,w}([-1,1]^d)$, $1 \le p < \infty$ and define the operator

$$L_n(f,t) = (\Omega)^{\frac{1}{n}} \int_{[-1,1]^d} \dots \int_{[-1,1]^d} f(t+t_1+\cdots+t_k)dt \;,\; \Omega = \frac{1}{k} \text{ Then}$$

$i-\; L_n(f,t) \ge 0$.

$ii - ||\, L_n(f,t)^{\,k}\, ||_{p,w,([-1,1]^d)} \le C_{k1}\omega \left(f,\frac{1}{n}\right)_{p,w,([-1,1]^d)}$.

$iii - ||f - L_n(f,\alpha)_{\,p,w}\, ||_{p,w,([-1,1]^d)} \le C_{k2}\omega \left(f,\frac{1}{n}\right)_{p,w,([-1,1]^d)}$.

$iv - ||f - L_n(f,\alpha)_{\,p,w}\, ||_{p,w,([-1,1]^d)} \le C_{k3}\omega \left(f',\frac{1}{n}\right)_{p,w,([-1,1]^d)}$.

Proof :

$$i - L_n(f,t) = (\Omega)^{\frac{1}{n}} \int_{[-1,1]^d} \dots \int_{[-1,1]^d} f(t+t_1+\cdots+t_k)dt$$

$$\int_{[-1,1]^d} \dots \int_{[-1,1]^d} f(t+t_1+\cdots+t_k)dt \ge .$$

Thus, $L_n(f,t) \ge 0$

$$ii - L_n(f,t) = (\Omega)^{\frac{1}{n}} \int_X \dots \int_X f(t+t_1+\cdots+t_k)dt$$

$$||\, L_n(f,t)||_{\,p,w,([-1,1]^d)} \le ||\, L_n(f,t) - g(t) + g(t)||_{\,p,w,([-1,1]^d)}$$

$$\le \; ||L_n(f,t) - g(t)||_{\,p,w,([-1,1]^d)} + \|g(t)\|_{p,w,([-1,1]^d)}$$

$$\le Max\,(c) \quad ||\, \Delta_\delta^1 f(t)\, ||_{p,w,([-1,1]^d)}$$

$$\le Max\,(c) \;\; Sup \;||\, \Delta_\delta^1 f(t)\, ||_{p,w,([-1,1]^d)}$$

$$\le Max\,(C) \quad \omega \quad (f,\frac{1}{n})_{\;p,w,([-1,1]^d)}$$

$$iii - ||f - L_n(f,\alpha)||_{\,p,w,([-1,1]^d)} = \left(\int_x |\, f_{(x)} - \; L_n\,(f\,)\, w(x)|^{\,p}\, dx\right)^{\frac{1}{p}}$$

$$\le Sup \left(\int_x |\, f_{(x)} - \; L_n\,(f\,)\, w(x)|\, Pdx\right)^{\frac{1}{p}} =$$

$$\le \quad C_{\,k2}\, Sup \;\; (\|\Delta^{\,1}_{\,\delta} f_{(.)}\|_{\,p,w,([-1,1]^d)}$$

$$\le \quad C_{\,k2}\, \omega \;\; (f,\frac{1}{n})_{\;p,w,([-1,1]^d)}$$

iv- from (iii) by clear

$$||f - \; L_n(f)||_{\,p,w,([-1,1]^d)} \le \quad C_{k2} \;\; \omega \;\; (f,\frac{1}{n})_{\;p,w,([-1,1]^d)}$$

from lemma (2.6)

$$||f - \; L_n(f)||_{\,p,w,([-1,1]^d)} \le \quad C_{k3} \;\; \omega \;\; (f',\frac{1}{n})_{\;p,w,([-1,1]^d)}$$

Theorem 3.2 : Let $f \in L_{p,w}([-1,1]^d)$, $1 \le p < \infty$ and there exist class of algebraic Polynomials $\{G_{\,k}\}_{\,k=1}^{\,n}$ satisfies

$$\int_x G_{\,k}(x)\; dx = 1$$

$$\|G_k\|_{\,p,w} \le C_k \;, \quad Where\; C_k \; is\; positive\; constant\;,$$

$if \;\; f \;\; satisfies\; Theorem$ 3.1

$\lambda = f_n(f,\delta)$, $\delta = \frac{1}{n}$ and $Cf_{\,n} = \int_{-\frac{1}{2}}^{\frac{1}{2}} \lambda(x) G_k(y - x)dx$. $Then$

1- $\|f - f^*\|_{\,p,w,([-1,1]^d)} \le C_k\, \omega_k\,(f,\delta)_{\,p,w}$

1- $\|f - f^*\|_{\,p,w,([-1,1]^d)} \le C_k\, \omega_k(f',\delta)_{\,p,w}$

Proof :

Let $G_{\,k} = G_{k+1} = \cdots = G_{4k-1} = \frac{1}{2}$, $n \ge 2k$,

Let $G_{\,4n-4k}(x) = C_n\left[(p_{2n}(x)\; / x^2 - t_{1,2n}^2\,) \dots (x^2 - t_{k,2n}^2)\right]^{\,2} \dots \dots (1)$

Where $P_{\,2n}$ is algebriac Polynomial of degree $2n$ and $t_{1,2n}, \dots, t_{n,2n}$ are its positive Zeros in creasing order. Define the remaining $G_{\,n}$ with the relation

$$G_{\,4n+1} = G_{\,4n+2} = G_{\,4n+3} = G_{\,4n} \;\;,\quad n \ge k$$

We have

$$C_{\,1}\, n^{-1} \le t_{1,2n} \le \cdots \le t_{k,2n} \le C_2\, n^{-1}\,, \quad n > k \dots \dots (2)$$

By using $\|P_n\|_{\,p,w} = 1$

Let last equality being a consequence of starlings formula (1)and(2) to gather imply

$$G_{\,4n-4k}(0) \ge C\, c_n\, n^{4k-1} \;,\; n \ge 2k \;\dots\dots\dots\dots\dots (3)$$

Write

$$1 = \int_{-1}^{1} G_{\,4n-4k}(x)\; dx = \sum_{k=-n}^{n} B_k\;(2n+1) G_{\,4n-4k}(t_{k,2n+1})$$

Where $B_k\;(2n+1)$ are the weights of Gaussian quadrature formula , exact for Polynomials of degree 4n+1 , with nodes The zeros of the Legendre of degree 2n+1 Therefore

$$1 \ge B_0\;(2n+1) G_{\,4n-4k}(0)$$

Since $B_0\;(2n+1) = \pi(1+0(1)) \;/\; (2n+1)$

Then

$$G_{\,4n-4k}(0) \le C.n \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots (4)$$

From (3) and (4) imply $C_n \le C_n^{2-4k}$

By using The definition of the G_n and (2) implies

$$\|G_n\|_{p,w,([-1,1]^d)} \leq C_n^{2-4k}$$

Again let $n \geq 2k$ $X^{2k} G_{4n-4k}(x)$ *is polynomial of degree* $4n-2k$

Therefore for $i = 1, 2, \ldots\ldots\ldots, k$

$$Hi = \int_{-1}^{1} a^{2i} G_{4n-4k}(x)dx$$
$$= 2\sum b_{j,2n}^{2i} B_j(2n) G_{4n-4k}(b_{j,2n})$$

Where $B_j(2n)$ are the weights of the Gaussian quadrature formula , exact for polynomials of degree 4n-1 , with nodes The zeros of the Legendre polynomial of degree 2n .

Since G_{4n-4k} has zeros at $t_{k+1,\ 2n}, \ldots\ldots, t_{n,2n}$

Then $Hi = 2\sum_{j=1}^{k} t_{j,2n}^{2i} B_j(2n) G_{4n-4k}(t_{j,2n})$

Since G_{4n-4k} *has alocal maximum on* $[-t_{k+1,2n},\ t_{k+1,2n}]$

At zero. Then

$$B_j(2n) \leq \frac{\pi}{2n}\left(1 + 0(1)\right), j = 1,2, \ldots\ldots, k$$

From (2) and (4)

The definition of the G_{n1} imply

$$\int_{-1}^{1} x^{2i} G_n(x)dx \leq C_n^{-2i}, \qquad i = 1,2, \ldots,\ n \geq k \ \ldots\ldots (5)$$

By using (5) we get

$$1 - F_n(1,t) = \int_{-1}^{1} G_n(x)dx - \int_{-\frac{1}{2}}^{\frac{1}{2}} G_n(x)dx \leq$$
$$2\int_{\frac{1}{4}}^{1} G_n(x)dx \leq C_n^{2-4k}$$
$$F_n\left((x,t)^{2i}, t\right) = \int_{-\frac{1}{2}}^{\frac{1}{2}} (x-t)^{2i} G_n(x-t)dx$$
$$= \int_{-\frac{1}{2}}^{\frac{1}{2}} x^{2i} G_n(x)dx$$
$$\leq \int_{-1}^{1} (x)^{2i} G_n(x)dx$$

And applying (5) ,

$$F_n\left((x-t)^{2i}, t\right) \leq C^{-2n}, \qquad i = 1,2 \ldots, k \quad \ldots (6)$$

$$F_n(|x-t|^k, t) \leq \int_{-1}^{1} |x|^k G_n(x)dx$$
$$\leq \left[\int_{-1}^{1} x^{2k} G_n(x)dx\right]^{\frac{1}{2}} \leq C_n^{-k}$$

By using The Schwartz inequality and (6) for i is odd , Then

$$\left|F_n\left((x-t)^i, t\right)\right| = \left|\int_{-\frac{1}{2}}^{\frac{1}{2}} x^i G_n(x)dx\right|$$
$$\leq 2\int_{-\frac{1}{4}}^{1} x^i G_n(x)dx$$

Since G_n is even , Then

$$\left|F_n\left((x-t)^i, x\right)\right| \leq C_n^{2-4k}, i = 1,3,5 \ldots$$

If $f \in \left[-\frac{1}{2}, \frac{1}{2}\right]$, Then Taylor's theorem gives

$$\lambda(x) = \left[\sum_{i=0}^{k-1} \frac{\lambda^{(i)}{}_{(t)}(x-t)^i}{i!}\right] + \frac{1}{(k-1)!}\int_t^x \lambda^{(k)}(u)(x-u)^{k-1} du$$

Since the last term on the right hand side is bounded in modulus by

$\left(\frac{1}{k!}\right)|x-t|^k . \|\lambda^{(k)}\|_{p,w}$

$$|F_n(\lambda,t) - \lambda(t)| \leq |\lambda(t)||\lambda - F_n(\lambda)| + \sum_{i=1}^{k-1} \frac{|\lambda^{(i)}(t)|}{i!} . \left|F_n(x-t)^i, t\right| +$$
$$\left(\frac{1}{k!}\right) . \|\lambda^{(k)}\|_{p,w} F_n\left(|u|^k, f(u)\right)$$

Thus $\| F_n(\lambda,t) - \lambda(t)\|_{p,w} \leq C . \|\lambda\|_{p,w,([-1,1]^d)} +$

$$\sum_{i=1}^{k-1} \frac{\|\lambda^{(i)}\|_{p,w}}{i!} . \ \|F_n(u)^i, f(u)\|_{p,w,([-1,1]^d)} +$$
$$\left(\frac{1}{k!}\right) \|\lambda^{(k)}\|_{p,w} . \|F_n\left((u)^k, f(u)\right)\|_{p,w,([-1,1]^d)}$$

$\leq Max\ (c) . w_k\ (f,\delta)_{p,w,([-1,1]^d)}$.

We have , from Theorem 3.1

$\|f\| \leq C\ \|f'\|$ implies

$$\|f - f^*\| \leq Max\ (c)\ \ \omega_k\ (f',\delta)_\alpha$$

Theorem 3.3: Let $\in L_{p,w}([-1,1]^d)$, $1 \leq p < \infty$ and $r \in \mathbb{N}$. Then

$$E_r(f,\delta)_{p,w} \leq Max(c)\ \ \omega_r(f,\delta)_{p,w}$$

Proof:- Let $f = \alpha - \beta$ such that

$$\alpha(x) = \beta(-1) + 2(\beta(1) - \beta(-1))(x+1)$$

From of property of smoothness of modulus , we have

$$\omega_r(f,\delta)_{p,w,([-1,1]^d)} \leq C\omega_r(\alpha,\delta)_{p,w,([-1,1]^d)}$$

Theorem 3.1 and 3.2 apply to , writing

$$\overline{x_k}(\alpha) = \beta(x) + x_n^*(x_k(f))$$

Theorem 3.1 and 3.2 imply

$$\|f - \overline{x_n}(f)\|_{p,w,([-1,1]^d)} = (\int_X |\alpha(x) - x(f,x).w(x)|^p dx)^{\frac{1}{p}}$$

$$\leq (\int_{X^d} |\alpha(x) - g(x).w(x)|^p dx)^{\frac{1}{p}}$$

$$+ \int_{X^d} |g(x) - x(f,x).w(x)|^p dx)^{\frac{1}{p}}$$

Let $g = x(f)$ Then

$$\overline{x_n}(f) = \beta(x) + x^*(g) = \beta(x) + \int_{X^d} g(t).\lambda_n(t-x)dt$$

$$\overline{x_n}(\alpha,x)' = \beta'(x) + \int_X g(t).-\lambda'_n(t-x)dt$$

$$= \beta'(x) + [-g(t)\lambda_n(t-x)]_{X^d} + \int_{X^d} g(t)\lambda_n(t-x)dt$$

$r \geq 2$ alternate differen and integration by parts yield :

$$\overline{x_k}(\alpha,x)^r = (-1)^n \left[\sum_{j=0}^{r-1} (-1)^j g^j(t)\,\lambda_n(r-1-j)(t-x)\right]_{X^d}^{\frac{1}{2}}$$

$$+ \int_{X^d} g^r(t)\lambda_n(t-x)dt$$

$$= \gamma(x) + \int_{X^d} g^r(t)\lambda_n(t-x)dt$$

From Theorem 3.1 , we get

$$\|f(\alpha) - r)\|_{p,w,([-1,1]^d)} \leq \|\Delta_\alpha^r f\|_{p,w,([-1,1]^d)}$$

$$\|f(\alpha) - r\|_{p,w,([-1,1]^d)} \leq sup\|\Delta_\alpha^r f\|_{p,w,([-1,1]^d)}$$

$$E_r(f,\delta)_{p,w,([-1,1]^d)} \leq Max(c)\ \omega_r(f,\delta)_{p,w,([-1,1]^d)}\ .$$

4. Conclusion

In this work, we managed of prove some properties modulus of smoothness which need it in main results proofs and we can found the degree of best monotone multi-approximation of unbounded functions by using some types of linear operators and algebraic polynomials.

1- G.G. Lorentz, (1986) , Approximation of functions , 2nd. Ed. , Chelsea, New York.

2- T. Nishishiraho, (1996), The degree of best approximation in Banach space , Tohoku Math. Journal , 46 ,p.p. 525-544.

3- I.A. Shevehak,(1992) , Approximation of monotone functions by monotone polynomials , Mat. Sb. 183,no5, p.p. 63-78 , (Russian , with Russian summary); English transl. , Russian Acad. Sci. Math 76 (1993), no1, p.p.51-64.

4- D.Leviatan ,(2002), co convex approximation ,J. Approximation Theory 118 ,p.p. 20-65 .

5- D.Leviatan and I.L. petreva,(2017), Interpolatory estimates in monotone piecewise polynomial approximation , J. Approximation Theory ,223, p.p. 1-8.

6- V.K. Dzyubenko, D.Leviatan and I.A. shevchuk, (2014), pointwise estimates of co-convex approximation, Jaen, J. Approx. , 6,p.p.261-295.

7- K.A. Kopotun, D.Leviatan and I.A. shevchuk ,(2011) , uniform and pointwise shape preserving Approximation by algebraic polynomials , surv – Approx. Theory 6 ,p.p. 24-74.

8- K.Kopotun ,(2015), , weighted moduli of smoothness of k. monotone functions and applications J.Approx. Theory ,192, p.p. 102-131 .

9- K.Kopotun, D.Leviatan and I.A. Shevchuk ,(2019), , on weighted approximation with Jacobi weights,J. Approx. Theory, 237 ,p.p. 96-112.

10-, on some properties of moduli of smoothness with Jacobi weights, Topics in classical and modern Analysis , Applied and numerical Harmonic analysis ,Springer ,2019, p.p. 19-31.

11-O.Vlasiuk, (2015), on the degree of piecewise shape-presening approximation by polynomials, J.Approx. Theory 89,p.p. 67-75.

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أفضل تقريب متعدد للدوال الغير مقيدة بواسطة مقياس النعومة

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الخلاصة:

الغرض من هذا البحث هو مناقشة درجة افضل تقريب افضل تقريب رتيب متعدد للدوال الغير مقيدة في الفضاء الموزون بشروط مقياس النعومة بواسطة بعض المؤثرات الخطية الجبرية وكذلك عرض براهين بعض خواص مقياس النعومة .