P- ISSN 1991-8941 E-ISSN 2706-6703
2022,(16), (2):92– 98
Journal of University of Anbar for Pure Science (JUAPS) Open Access

On Better Approximation of the Squared Bernstein Polynomials

Rafah F. Katham and Ali J. Mohammad

Department of Mathematics, College of Education for Pure Sciences, University of Basrah, Basrah, Iraq.
Email: rafah.fouad@gmail.com
Email: alijasmoh@gmail.com, ali.mohammad@uobasrah.edu.iq

ARTICLE INFO

Received: 03 / 09 /2022
Accepted: 15 / 11 / 2022
Available online: 22 / 12 / 2022

DOI: 10.37652/juaps.2022.176478

Keywords:
Squared B ernstein polynomial,
Modulus of Smoothness,
Quantitative Voronovskaja theorem,
and Grüss-Voronovskaja theorem,

ABSTRACT

The present paper is defined a new better approximation of the squared Bernstein polynomials. This better approximation has been built on a positive function τ defined on the interval [0,1] which has some properties. First, the moderate uniform convergence theorem for a sequence of linear positive operators (the generalization of the Korovkin theorem) of these polynomials is improved. Then, the rate of convergence of these polynomials corresponding to the first and second modulus of continuity and Ditzian- Totik modulus of smoothness is given. Also, the quantitative Voronovskaja and the Grüss- Voronovskaja theorems are discussed. Finally, some numerically applied for these polynomials are given by choosing a test function$f$ and two different τ functions show the effect of the different chosen functions τ. It turns the new better approximation of the squared Bernstein polynomials gives us a better numerical result than the numerical results of both the classical Bernstein polynomials and the squared Bernstein polynomials. **MSC 2010.** 41A10, 41A25, 41A36.

1-Introduction

For $x \in [0,1]$ and a function $f \in C[0,1]$, the well-known Bernstein polynomials are defined as

$$B_n(f,x)=\sum_{k=0}^{n} b_{n,k}(x)\, f\left(\frac{k}{n}\right), \qquad (1.1)$$

where $b_n(x)=\binom{n}{k}x^k(1-x)^{n-k}$.

Voronovskaja showed that the polynomials $B_n(f,x)$ satisfied the property $\lim_{n\to\infty} n\{B_n(f;x)-f(x)\}=\frac{x(1-x)}{n}f''(x)$. So, to order of approximation of the polynomials $B_n(f,x)$ is $O(n^{-1})$.

a new modification of the sequence Bernstein polynomials introduced by King for $f \to B_n(f)\circ r_n(x)$, $r_n(x)\in C[0,1]$ that is preserved two functions 1 and x^2 as[1]

$$V_n(f;x)=\sum_{k=0}^{n}\binom{n}{k}\left(r_n(x)\right)^k\left(1-r_n(x)\right)^{n-k} f\left(\frac{k}{n}\right). \qquad (1.2)$$

Where

$$r_n(x) = \begin{cases} x^2 & ; \quad n=1 \\ -\frac{1}{2(n-1)}+\sqrt{\left(\frac{n}{n-1}\right)x^2+\frac{1}{4(n-1)^2}} & ; \quad n=2,3,\dots. \end{cases}$$

Morales, Garranchoa, and Raşa present a Bernstein-type polynomial defines for a function $f \in C[0,1]$ by[2]

$$B_{n,\tau}(f;x)=\sum_{k=0}^{n} b_{n,k}(x)\, f\circ\tau^{-1}\left(\frac{k}{n}\right), \quad x \in [0,1]. \qquad (1.3)$$

Ioan and Mircea define squared Bernstein polynomial, for $Q_n: C[0,1]\to C[0,1]$.as[3]

$$Q_n(f;x)=\frac{\sum_{k=0}^{n} b_{n,k}^2(x) f(x)}{\sum_{k=0}^{n} b_{n,k}^2(x)} \qquad (1.4)$$

Abdul Samad and Mohammad studied about squared Bernstein polynomials and deduced the convergence of $Q_n(f;x)$ and calculate the recurrence relation for the m-th order moment[4,5].

, Long and Zeng indicate a new a modification of Bernstein sequence depended on λ, then, Muhammad and Jaber give a modification and d expressed the Voronovaskaja-type for the λ -Bernstein sequence[6].

*Corresponding author at : Department of Mathematics, College of Education for Pure Sciences, University of Basrah, Basrah, Iraq. ;ORCID:https://orcid.org/0000-0000-00000-;Tel:
E-mail address: rafah.fouad@gmail.com

2. Definitions and Results.

This section presents several definitions and lemma that help us.

Definition 2.1.

The p-th order of modulus of smoothness for $r \in \mathbb{R}$ is given by[7]:

$$\omega_p(f;r) = \sup_{\substack{|\delta|\le r\\ x+j\delta\in I}} \left\{ \left| \sum_{j=0}^{p} (-1)^{p-j} \binom{p}{j} f(x+j\delta) \right| \right\}.$$

The p-th order of modulus of smoothness has the following properties: [11]

setting $f \in C^q[0,\ 1]$, and for $0 < r < \frac{1}{2}$ there exist functions $v \in C^{q+2}[0,\ 1]$ such that

(i) $\|f^{(q)} - v^{(q)}\| \le \frac{3}{4}\omega_2(f^{(q)};r)$;

(ii) $\|v^{(q+1)}\| \le \frac{5}{h}\omega_1(f^{(q)};r)$;

(iii) $\|v^{(q+2)}\| \le \frac{3}{2r^2}\omega_2(f^{(q)};r)$.

The polynomials $B_{n,\tau}(f,x)$ have the following properties[2]

(i)$B_{n,\tau}(1;x) = 1$;

(ii)$B_{n,\tau}(\tau;x) = \tau$;

(iii)$B_{n,\tau}(\tau^2;x) = \left(1-\frac{1}{n}\right)(\tau(x))^2 + \frac{\tau(x)}{n}$.

Properties of polynomial $Q_n(f;x)$, let's define[3]

$$Q^2_{n,m}(x) = \frac{\sum_{k=0}^{n} b^2_{n,k}(x)K^m}{\sum_{k=0}^{n} b^2_{n,k}(x)}$$

(i) $Q^2_{n,0}(x) = 1$;

(ii) $Q^2_{n,1}(x) = \frac{2n+1}{2}x - \frac{1}{4} + o(1)$;

(iii) $Q^2_{n,2}(x) = \frac{4n^2+2n}{4}x^2 - \frac{3}{16n^2} + o(1)$.

3. Main Work.

For τ is a continuous positive function with the properties $\tau(1) = 1, \tau(0) = 0, \forall x \in [0,1]$ and differentiable for infinite times. The polynomials are a combinate of the squared Bernstein polynomials with a positive function τ to get a better approximation which is defined as follows

$$SB_{n,\tau}(f;x) = \sum_{k=0}^{n} \frac{b^2_{n,k}(\tau(x))(f\circ\tau^{-1}(x))}{\sum_{k=0}^{n} b^2_{n,k}(\tau(x))}.$$

To study the convergence of the sequence $SB_{n,\tau}(f;x)$, one needs to introduce the following lemma.

Lemma 3.1.

For $x \in [0,1]$, $e_i(\tau) = \tau^i$, $i = 0,1,2$. One get.

(i) $SB_{n,\tau}(\tau^0;x) = 1$;

(ii) $SB_{n,\tau}(\tau^1;x) = \frac{2x-1}{4}\tau(x) + \frac{1}{4} + o(1)$;

(iii) $SB_{n,\tau}(\tau^2;x) = \frac{4n^2+2n}{4}\tau(x)^2 + \frac{1}{16} + o(1)$.

Proof.

Using the reference by the direct evaluation, one can get directly the proof of this Lemma. ■

Theorem 3.1.

Let $f \in C[0,1]$. The polynomial $SB_{n,\tau}(f;x)$ converges to f on the [0, 1] uniformly.

Proof.

Using the above properties in Lemma 3.1, the convergence of the polynomials $SB_n(f;x)$ to the function $f(x)$ is proved. ■

Lemma 3.2.

Let $f \in C[0,1]$. We have $\|SB_{n,\tau}(f;x)\| \le \|f\|$. Since the $\|.\|$ is the sup-norm on [0, 1].

Proof.

By using the properties $SB_{n,\tau}(\tau^0;X) = 1$

We get that $\|SB_{n,\tau}(f;x)\| \le \|f\circ\tau^{-1}(x)\| SB_{n,\tau}\tau^0 = \|f\|$

then$\|SB_{n,\tau}(f;x)\| \le \|f\|$.

Definition 3.1.

The m-th order moment, where $m \in \{0,1,\ldots\}$, for the polynomials $SB_{n,\tau}(.;x)$ is denoted and defined by $U_{n,m,\tau}(x) = SB_{n,\tau}\left(\left(\tau(t)-\tau(x)\right)^m;x\right)$.

Lemma 3.3.

The central moment polynomial has

(i) $U_{n,0,\tau}(x) = 0$;

(ii) $U_{n,1,\tau}(x) \simeq \frac{2\tau(x)-1}{2n}$;

(iii) $U_{n,2,\tau}(x) \simeq \frac{(1-\tau(x))(2n\tau(x)+1)}{4n^2}$;

(iv) $U_{n,3,\tau}(x) \simeq \frac{(\tau(x)-1)(2n\tau^2(x)-4n\tau(x)-1)}{8n^3}$;

(v) $U_{n,4,\tau}(x) \simeq \frac{(\tau(x)-1)(12n^2\tau^3(x)-12n^2\tau^2(x)+4n\tau^3(x)-6nx-1)}{16n^4}$.

Proof.

From Lemma 3.1 and the Definition 3.1.one gets the consequences result.

Lemma 3.4.

For $x \in [0,1]$ and $n \in N$ then $\frac{U_{n,4,\tau}(x)}{U_{n,2,\tau}(x)} \le \frac{3\tau^2(x)}{4n^2}$.

Proof.

$\frac{U_{n,4,\tau}(x)}{U_{n,2,\tau}(x)} = \frac{(\tau(x)-1)(12n^2\tau^3(x)-12n^2\tau^2(x)+4n\tau^3(x)-6n\tau(x)-1)}{16n^4} \cdot \frac{2n^2}{(1-\tau(x))(2n\tau(x)+1)}$

.

$\leq \frac{6n\tau^2(x)}{8n^2} = \frac{3\tau^2(x)}{4n^2}$. ■

Theorem. 3.2.

For $f \in C^2[0,1]$ and for all $x \in [0,1]$. Then

$$\left|SB_{n,\tau}(f;x) - f(x)\right| \leq \frac{(1-\tau(x))(2n\tau(x)+1)}{4n^2} \cdot \left(\frac{\|f''\|}{\alpha^2} + \frac{\|f'\|\|\tau''\|}{\alpha^3}\right).$$

Proof.

Setting Taylor's expansion gets

$f(y) = (f \circ \tau^{-1})\tau(x)$

$$f(y) = (f \circ \tau^{-1})\tau(x) + (\tau(y) - \tau(x))(f \circ \tau^{-1})'\tau(x) + \int_{\tau(x)}^{\tau(y)} (f \circ \tau^{-1})'(u)(\tau(y) - u)du.$$

Since

$$\int_{\tau(x)}^{\tau(t)} (f \circ \tau^{-1})'(u)(\tau(y) - u)du = \int_{\tau(x)}^{\tau(y)} (\tau(y) - u)\frac{f''(\tau^{-1}(u))}{(\tau'(\tau^{-1}(u)))^2}du - \int_{\tau(x)}^{\tau(y)} (\tau(y) - u)\frac{f'(\tau^{-1}(u))\tau''(\tau^{-1}(u))}{(\tau'(\tau^{-1}(u)))^3}du$$

Applying $SB_{n,\tau}(f;x)$ on both sides,

$$\left|SB_{n,\tau}(f;x)\right| = f(x) + SB_{n,\tau}\left(\int_{\tau(x)}^{\tau(y)}(\tau(y) - u)\frac{f''\left(\tau^{-1}(u)\right)}{(\tau'(\tau^{-1}(u)))^2}du; x\right) - KB_{n,\tau,s}\left(\int_{\tau(x)}^{\tau(y)}(\tau(y) - u)\frac{f'\left(\tau^{-1}(u)\right)\tau''\left(\tau^{-1}(u)\right)}{(\tau'(\tau^{-1}(u)))^3}du; x\right)$$

For simplicity $\alpha^2 = (\tau'(\tau^{-1}(u)))^2, \alpha^3 = (\tau'(\tau^{-1}(u)))^3$

$$\left|SB_{n,\tau}(f;x) - f(x)\right| \leq SB_{n,\tau}\left((\tau(y) - \tau(x))^2; x\right) \times \left(\frac{\|f''\|}{\alpha^2} + \frac{\|f'\|\|\tau''\|}{\alpha^3}\right)$$

$$\leq \frac{(1-\tau(x))(2n\tau(x)+1)}{4n^2}\left(\frac{\|f''\|}{\alpha^2} + \frac{\|f'\|\|\tau''\|}{\alpha^3}\right). ■$$

Theorem 3.3.

Form the definition 2.1 of modulus of continuity and $f \in C[0,1]$. Then $\left|SB_{n,\tau}(f;x) - f(x)\right| \leq \lambda_n(x)\omega_1\left((f \circ \tau^{-1}); \lambda_n(x)\right)$, for $x \in [0,1]$. $\sqrt{U_{n,2,\tau}(x)} = \lambda_n(x)$

Proof.

Let $f(y) = (f \circ \tau^{-1})\tau(x)$, the using Taylor's expansion

$$f(y) = (f \circ \tau^{-1})\tau(x) + (\tau(y) - \tau(x))(f \circ \tau^{-1})'\tau(x) + \int_{\tau(x)}^{\tau(y)} (f \circ \tau^{-1})'(u)(\tau(y) - u)du.$$

Applying $SB_{n,\tau}(.;x)$ we get

$$SB_{n,\tau}(f;x) - f(x) = SB_{n,\tau}\left(\int_{\tau(x)}^{\tau(y)} (f \circ \tau^{-1})'(u)(\tau(y) - u)du; x\right)$$

From Definition 2.1, we have

$$\left|B_{n,\tau,s}(f;x) - f(x)\right| \leq \sqrt{U_{n,2,\tau}(x)}\left\{1 + \frac{1}{h}\sqrt{U_{n,2,\tau}(x)}\right\}\omega_1((f \circ \tau^{-1}); h)$$

$$\leq \sqrt{U_{n,2,\tau}(x)}\omega_1\left((f \circ \tau^{-1}); h\right) + \sqrt{U_{n,2,\tau}(x)}\omega_1((f \circ \tau^{-1}); h)$$

$$\leq 2\sqrt{U_{n,2,\tau}(x)}\omega_1\left((f \circ \tau^{-1}); \sqrt{U_{n,2,\tau}(x)}\right)$$

Let $\sqrt{U_{n,2,\tau}(x)} = \lambda_n(x)$

$\left|B_{n,\tau,s}(f;x) - f(x)\right| = 2\lambda_n(x)\omega_1\left((f \circ \tau^{-1}); \lambda_n(x)\right)$. ■

Theorem 3.4.

For $f \in C^2[0,1]$, the approximation properties of modulus for $SB_{n,\tau}(f;x)$ verify

$$\left|SB_{n,\tau}(f;x) - f(x)\right| \leq \frac{3}{2}\omega_2((f \circ \tau^{-1}; h)$$

$$+\frac{(1-\tau(x))(2n\tau(x)+1)}{4n^2}\left(\frac{3}{2h^2\alpha^2}\omega_2((f\circ\tau^{-1};h)+\frac{5.\|\tau''\|}{h\alpha^3}\omega_1((f\circ\tau^{-1};h)\right).$$

Proof.

$$\left|SB_{n,\tau}(f;x)-f(x)\right| = \left|SB_{n,\tau}(f;x)+SB_{n,\tau}(g;x)-SB_{n,\tau}(g;x)+g(x)-g(x)-f(x)\right|$$
$$\leq \left|SB_{n,\tau}(f-g;x)\right|+\left|SB_{n,\tau}(g;x)-g(x)\right|+|g(x)-f(x)|$$

Using theorem 3.2

$$\leq 2\|f-g\|+\frac{(1-\tau(x))(2n\tau(x)+1)}{4n^2}\left(\frac{\|f''\|}{\alpha^2}+\frac{\|f'\|\|\tau''\|}{\alpha^3}\right)$$

For $h>0, h\leq\frac{1}{2}\ni g\in C^2[0,1]$, and

$\|f-g\|\leq\frac{3}{4}\omega_2(f;\ h),\ \|v'\|\leq\frac{5}{h}\omega_1(f;h)\ \ \|v''\|\leq\frac{3}{2h^2}\omega_2(f;h)$

Then

$$\left|SB_{n,\tau}(f;x)-f(x)\right| \leq\frac{3}{2}\omega_2(f;h)+\frac{(1-\tau(x))(2n\tau(x)+1)}{4n^2}\left(\frac{3}{2h^2\alpha^2}\omega_2(f;h)+\frac{5\|\tau''\|}{h\alpha^3}\omega_1(f;h)\right).\ \blacksquare$$

4. Voronovskaja-type theorem.

In this part, Quantitative Voronovskaja and Grüss-Voronovskaja[8] are proved which is one of the most important theorems to prove pointwise convergence.

Theorem 4.1.

For $f\in C^2[0,1]$ we have that

$$\left|SB_{n,\tau}(f;x)-f(x)-\frac{(1-\tau(x))(2n\tau(x)+1)}{4n^2}(f\circ\tau^1)''\tau(x)\right| \leq\frac{1}{2}U_{n,2,\tau}(x)\breve{\omega}\left(f'';\frac{1}{3}\sqrt{\frac{3\tau^2(x)}{4n^2}}\right),$$

where

$$\breve{\omega}(f'';\vartheta) := \begin{cases}\sup\frac{(\vartheta-u)\omega(f;v)+(v-\vartheta)\omega(f,u)}{v-u}, \text{if } 0\leq\vartheta\leq b-a\\ \omega(f;b-a), \text{if } \vartheta b-a\end{cases},$$

is the least concave majorant.

Proof.

By the Quantitative Voronovskaja theorem[8]

$\left|L_{n,}(f,x)-f(x)-f'(x)\mu_{n,1}(x)-\frac{1}{2}f''(x)\mu_{n,2}(x)\right|\leq \frac{1}{2}\mu_{n,2}(x)\breve{\omega}\left(f'';\frac{1}{3}\sqrt{\frac{\mu_{n,4}(x)}{\mu_{n,2}(x)}}\right)$,

applying $B_{n,\tau}$, and using the values of $U_{n,1}(x), U_{n,2}(x)$, we get

$$\left|SB_{n,\tau}(f,x)-f(x)-f'(x)\frac{2\tau(x)-1}{2n}-\frac{1}{2}f''(x)U_{n,2}(x)\right| \leq\frac{1}{2}\frac{(1-\tau(x))(2n\tau(x)+1)}{4n^2}\breve{\omega}\left(f'';\frac{1}{3}\sqrt{\frac{3\tau^2(x)}{4n^2}}\right).\ \blacksquare$$

Theorem 4.2.

For $f,g\in C^2[0,1]$, we get

$$\left|SB_{n,\tau}(fg;x)-SB_{n,\tau}(f;x)SB_{n,\tau}(g;x)-U_{n,2,\tau}(x)\frac{f'(x)g'(x)}{[\tau'(x)]^2}\right|$$
$$\leq\frac{1}{2}\frac{(1-\tau(x))(2n\tau(x)+1)}{4n^2}\breve{\omega}((fg\circ\tau^{-1})'';t_n)+\|g\|\breve{\omega}((fg\circ\tau^{-1})'';t_n)+\|f\|\breve{\omega}((fg\circ\tau^{-1})'';t_n)+\frac{1}{4}\frac{(1-\tau(x))(2n\tau(x)+1)}{4n^2}(x).$$

Proof.

Through using the decomposition formula[11],

$$SB_{n,\tau}(fg;x)-SB_{n,\tau}(f;x)SB_{n,\tau}(g;x)-U_{n,2,\tau}(x)\frac{f'(x)g'(x)}{[\tau'(x)]^2}$$

$= SB_{n,\tau}(fg;x)-(fg)(x)-\frac{1}{2}U_{n,2,\tau}(x)(fg\circ\tau^{-1})''\tau(x)-f(x)\left[SB_{n,\tau}(g;x)-g(x)-\frac{1}{2}U_{n,2,\tau}(x)(g\circ\tau^1)''\tau(x)\right]-g(x)\left[SB_{n,\tau}(f;x)-f(x)-\frac{1}{2}U_{n,2,\tau}(x)(f\circ\tau^1)''\tau(x)\right]+\left[f(x)-SB_{n,\tau}(f;x)\right]\left[SB_{n,\tau}(g;x)-g(x)\right]$

$$=|\gamma_1|-|\gamma_2|-|\gamma_3|+|\gamma_4|$$

from Theorem 4.1

$$|\gamma_1| \leq \frac{(1-\tau(x))(2n\tau(x)+1)}{4n^2}\breve{\omega}\left((fg\circ\tau^1)''; \frac{1}{3}\sqrt{\frac{3\tau^2(x)}{4n^2}}\right)$$

$$|\gamma_2| \leq \|f\|\frac{(1-\tau(x))(2n\tau(x)+1)}{4n^2}\breve{\omega}\left((g\circ\tau^1)''; \frac{1}{3}\sqrt{\frac{3\tau^2(x)}{4n^2}}\right)$$

$$|\gamma_3| \leq \|g\|\frac{(1-\tau(x))(2n\tau(x)+1)}{4n^2}\breve{\omega}\left((f\circ\tau^1)''; \frac{1}{3}\sqrt{\frac{3\tau^2(x)}{4n^2}}\right).$$

Since,

$$\left|SB_{n,\tau}(f;x)-f(x)\right| \leq \frac{1}{2}SB_{n,\tau}(f\circ\tau^1)''\tau(\epsilon)(\tau(t)-\tau(x))^2;x)$$

$$\leq \frac{1}{2}\|f\circ\tau^1\|U_{n,2,\tau}(x)$$

$$\leq \frac{(1-\tau(x))(2n\tau(x)+1)}{4n^2}\|f\circ\tau^1\| := I_n(f;\ x).$$

$$\left|SB_{n,\tau}(fg;x)-SB_{n,\tau}(f;x)SB_{n,\tau}(g;x)-U_{n,2,\tau}(x)\frac{f'(x)g'(x)}{[\tau'(x)]^2}\right|$$

$$\leq \frac{(1-\tau(x))(2n\tau(x)+1)}{4n^2}\breve{\omega}\left((fg\circ\tau^1)''; \frac{1}{3}\sqrt{\frac{3\tau^2(x)}{4n^2}}\right) + \|f\|\frac{(1-\tau(x))(2n\tau(x)+1)}{4n^2}\breve{\omega}\left((g\circ\tau^1)''; \frac{1}{3}\sqrt{\frac{3\tau^2(x)}{4n^2}}\right)$$

$$+\|g\|\frac{(1-\tau(x))(2n\tau(x)+1)}{4n^2}\breve{\omega}\left((f\circ\tau^1)''; \frac{1}{3}\sqrt{\frac{3\tau^2(x)}{4n^2}}\right)+I_n(f,x).I_n(g,x). \blacksquare$$

Corollary 4.1.

For $f\in C^1[0,1]$, one has

$$\left|SB_{n,\tau}(f,x)-f(x)\right| \leq \frac{(1-\tau(x))2\tau(x)}{\sqrt{n}}\omega_1\left((f\circ\tau^{-1}); \frac{(1-\tau(x))2\tau(x)}{\sqrt{n}}\right).$$

5. Numerical examples.

To enhance the work, we took two examples in which we applied these polynomials for different functions and presented the results in the following figures.

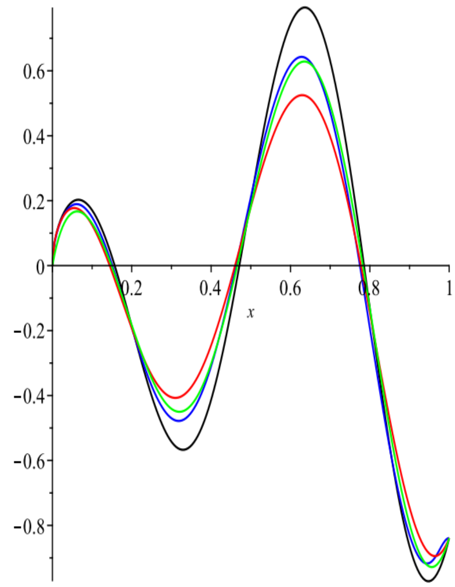

Fig.1: $f=\sqrt{x}\cos(8x), \tau(x)=\sqrt{x}$
black=$f(x)$ blue= $SB_{n,x,}(f,x)$
red=$B_{n,\tau}(f,x)$ green=$B_n(f,x)$

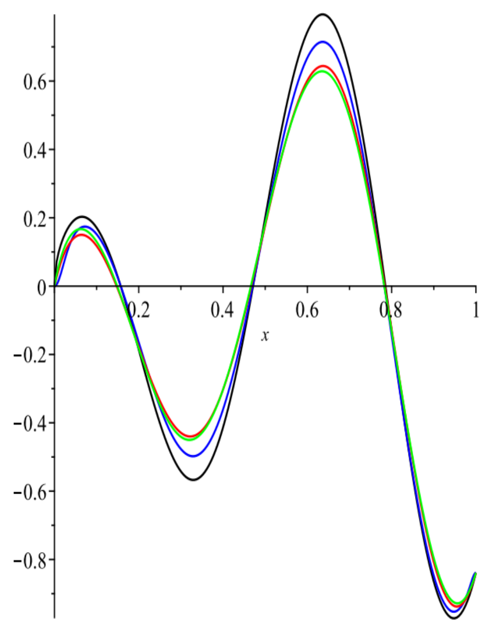

Fig.2: $f=\sqrt{x}\cos(8x), \tau(x)=\frac{x^2+2x}{3}$
black=$f(x)$ blue= $SB_{n,x,}(f,x)$
red=$B_{n,\tau}(f,x)$ green=$B_n(f,x)$

The following figures show the error result for the two examples above, Fig.3 for Fig.1 And Fig.2 for Ex 5.1.

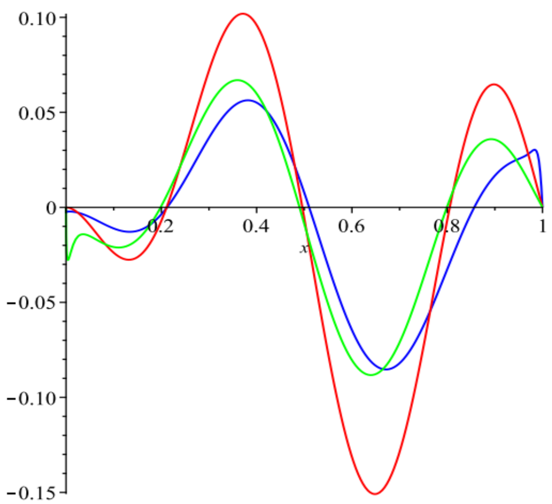

Fig.3: $f = \sqrt{x}\, cos(8x)\, , \tau(x) = \frac{x^2+2x}{3}$
blue= $SB_{n,x,}(f,x)$
red=$B_{n,\tau}(f,x)$ green=$B_n(f,x)$

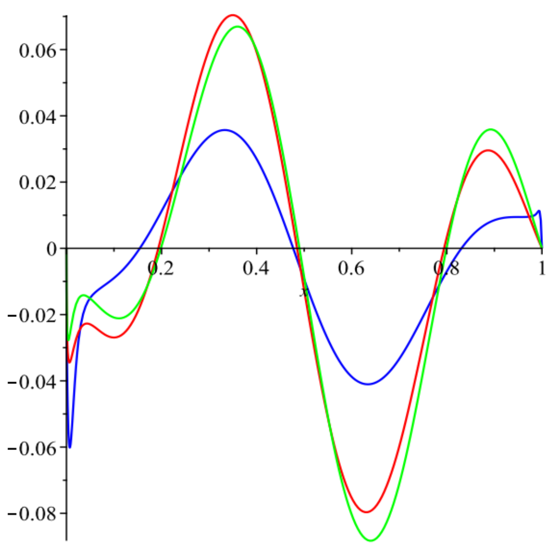

Fig.4:$f = \sqrt{x}\, cos(8x)\, , \tau(x) = \frac{x^2+2x}{3}$
blue= $SB_{n,x,}(f,x)$
red=$B_{n,\tau}(f,x)$ green=$B_n(f,x)$

6. Conclusion.

This paper has established the polynomials $SB_{n,\tau}(f,x)$ which are from the squared Bernstein polynomials. Also, these polynomials have studied the rate of convergence by the rate of modulus of smoothness. Next, the results work is supported with a numerical example of these polynomials which explained the effect of the function $\tau(x)$ and the squared to Bernstein polynomial.

References.

[1] J.P. King, Positive linear operators which preserve x2, Acta Math. Hungar, 99 (3) (2003) 203–208.
[2] D. Cárdenas-Morales a, P. Garranchoa, I. Raşa b, Bernstein-type operators which preserve polynomials. Computers and Mathematics with Applications 62 (2011) 158–163.
[3] I. Gavrea, M. Ivan, On a new sequence of positive linear operators related to squared Bernstein polynomials, Positivity. 21 (2017) 911–917.
[4] I. A. Abdul Samad, Ali J. Mohammad, The r-th Powers of the Rational Bernstein Polynomials.Basrah Journal of Science 39(2)(2021) 179-191.
[5] Ali Jassim Mohammad, Asma Jaber, On Some Approximation Properties for a Sequence of - Bernstein Type Operators. Iraqi Journal of Science, 2021, Vol. 62, No. 5, pp: 1666-1674.
[6] Ali J.Mohammad , Amal K.Hassan, Simultaneous Approximation by a New Sequence of Integral Type Based on Two Parameters. Iraqi Journal of Science, 2021, Vol. 62, No. 5, pp: 1666-1674.
[7] S.A. Mohiuddine, T. Acar, M.G. Alghamdi, Genuine modified Bernstein-Durrmeyer operators. J. Inequal. Appl. 104 (2018) 1-10.
[8] SG. Gal, H Gonska, Grüss and Grüss-Voronovskaya-type estimates for some Bernstein-type polynomials of real and complex variables. Jaen J Approx.7(1) (2015) 97-122.

P- ISSN 1991-8941 E-ISSN 2706-6703 2022,(16), (2):92– 98 *Journal of University of Anbar for Pure Science (JUAPS)* Open Access

حول أفضل تقريب لمتعددات حدود برنشتاين التربيعية

رفاه فؤاد كاظم، علي جاسم محمد

جامعة البصرة– كلية التربية للعلوم الصرفة– قسم الرياضيات

جامعة البصرة– كلية التربية للعلوم الصرفة– قسم الرياضيات

Email: rafah.fouad@gmail.com

Email: alijasmoh@gmail.com, ali.mohammad@uobasrah.edu.iq

المستخلص.

في هذا البحث تم تعريف أفضل تقريب جديد لمتعددات حدود برنشتاين التربيعية. حيث انه تم بناء أفضل تقريب لمتعددة حدود برنشتاين التربيعية بالاعتماد على دالة موجبة τ المعرفة بالفترة $[0,1]$ ولها بعض الخواص. أولا، تم برهان نظرية التقارب المنتظم المعدل للمتسلسلات الخطية الموجبة (نظرية كورفكن المعدلة) المتعددات الحدود برنشتاين التربيعية المعتمدة على الدالة τ . ثم تم إعطاء معدل تقارب متعددات الحدود بواسطة استعمال معامل الاستمرار الأول والثاني وديتانا– توتك لمعامل النعومة. وأيضا، تم مناقشة مقدار الخطأ عن طريق نظرتين هما نظريات فرونوفسكيا الكمي و كيرس فرونوفسكيا, وأخيرا, تم أعطاء بعض التطبيقات العددية على متعددات الحدود برنشتاين التربيعية المعتمدة على الدالة τ عن طريق اختيار دالة اختبار f واحدة ودالتين مختلفتين ل τ وبيان تأثير هذه الدالة τ على سرعة التقريب. أن أهمية البحث يبين أن أفضل تقريب لمتعددات حدود برنشتاين التربيعية تعطينا نتائج عددية أفضل من النتائج العددية لكلا من متعددات حدود برنشتاين الاعتيادية ومتعددات حدود برنشتاين التربيعية.