Approximate Solution of Emden-Fowler Equation Using the Galerkin Method

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ABSTRACT

The Emden-Fowler equation (E-F Eq.) used in mathematical with other science like physics, chemical physics and astrophysics, also this equation can be reduced to the Lane–Emden equation with specified function and used it in different sciences with mathematics. Many Authors study analytic and numerical methods to find the solution for this kind of the equations in the case linear or nonlinear one of these methods the homotopy-perturbation method.

In this work the approximate solution for generalized (E-F Eq.) in the second order ordinary differential equations was found by Galerkin method which is one of the weighted residual methods and do not need long time also use operator (linear or nonlinear) or differential operator in the any kind of the intervals and compared this solution with the exact solution by discuss the results from applying this method for homogeneous and nonhomogeneous equations and drawn the solutions in the same figure to illustrate the results.

Keywords: Emden-Fowler Equation, weighted residual method, Galerkin method.

1. Introduction:

In mathematics, in the field of applied mathematics, a great interest in numerical solutions emerged in the last century. Many methods appeared to solve initial and boundary value problems that are difficult to find an analytical solution. Some of these methods have gained wide fame either ease of use or for their efficiency compared to similar methods such as iterative methods like in [1], [2], Homotopy-perturbation method [3], Runge-Kutta method [4], Galerkin method[5],[6] which is the subject of study here. We used it to solve the Emden-Fowler equation (E-F Eq.).

In [7] and [8] study the initial value problems of (E-F Eq.),

\[ y'' + \frac{2}{x}y' + af(x)g(y) = 0, \quad x > 0, \] with \[ y(0) = \alpha \text{ and } y'(0) = 0, \] \[ \quad \ldots(*) \]

where \( \alpha \) is a fixed, \( f(x) \) and \( g(y) \) are given functions.

The problem of find the approximate solution for the equation (*) and compared that solution with the approximate solution by homotopy-perturbation method studied by [3].

2- Main Notions

\( u(x) \) is approximated solution in the weighted-residual method by

\[ \tilde{u}(x) \approx U_N(x) = \sum_{j=1}^{N} c_j \phi_j(x) + \phi_0(x) \]

The residual of the approximation is

\[ R = S(\sum_{j=1}^{N} c_j \phi_j(x) + \phi_0(x)) - h \neq 0 \]

where \( S \) is an operator linear or nonlinear.
The notion in the methods of weighted residuals as follows
\[ \int R(x)W_j dx = 0 \quad (j = 1, 2, \ldots, n) \]

Anywhere numeral weighted functions \( W_i = \) numeral of the constants \( c_j \), [9], [10].
In [6] Abd Almajeed surveys the steps of this method to find the approximate solution of the delay differential equations that in [11].

3- Application Examples

Example (1):
In [12] Ramos gave the exact solution \( n = m^4 - m^3 \) for the linear, nonhomogeneous (E-F.Eq.)
\[ n^* + \frac{8}{W} n' + m n - m^5 + m^4 - 44m^2 - 30m = 0 \]
\( n(0) = 0 \) and \( n'(0) = 0 \)

To find the approximate solution we apply the same steps in [6]
\[ R(m) = n^* + \frac{8}{W} n' + m n - m^5 + m^4 - 44m^2 - 30m \]

Suppose \( \tilde{n}(m) = a_0 + a_1 m + a_2 m^2 \). So from the conditions we get \( a_0 = 0 \)
\[ \tilde{n}'(m) = a_1 + 2a_2 m \] also from the conditions we get.
\[ \tilde{n}''(m) = 2a_2 \]

\[ R(m) = 18a_2 + 2m a_2 - m^5 + m^4 - 44m^2 - 30m \Rightarrow W = \frac{\partial R}{\partial a_2} = 18 + m^3 \]
\[ \int WR(m) dm = \int (18 + m^3)[18a_2 + a_2 m^3 - m^5 + m^4 - 44m^2 + 30m] dm \]

Implies after reduce above integral, \( a_2 = \frac{336689547}{47334880} \)
So the approximate solution is
\[ \tilde{n}(m) = \frac{336689547}{47334880} m^2 \]

Similarly we can apply those steps if we take other intervals.

Example (2):
Apply the steps in [6] to find the approximate solution for the nonlinear, homogeneous (E-F.Eq.)
\[ k^* + \frac{8}{m} k' + 18a k + 4a k \ln k = 0, \quad k(0) = 1 \quad \text{and} \quad k'(0) = 0 \]
The exact solution is \( k(m) = e^{-a m^2} \), [13].
As in [6], the steps for find the approximate solution are
\[ R(m) = k^* + \frac{8}{m} k' + 18a k + 4a k \ln k \]

Suppose \( \tilde{k}(m) = s_0 + s_1 m + s_2 m^2 \). So from the conditions we get \( s_0 = 1 \)

Also from the conditions we get \( s_1 = 0 \)
\[ \tilde{k}'(m) = s_2 \]
\[ R(m) = 18s_2 + 18a+18as_2m^2 + 4a ln(1+s_2m^2) + 4as_2m^2 ln(1+s_2m^2) \]
\[ W = \frac{\partial R}{\partial s_2} = 18 + 36a + \frac{8a m^2 + 4as_2m^2}{(1+s_2m^2)} \frac{2m}{(1+s_2m^2)} + 8a m \ln(1+s_2m^2) \]
\[ \int WR(m) dm = \int (18 + 36a + \frac{8a m^2 + 4as_2m^2}{(1+s_2m^2)} \frac{2m}{(1+s_2m^2)} + 8a m \ln(1+s_2m^2)) \]
\[ \left[ 18s_2 + 18a+18as_2m^2 + 4a ln(1+s_2m^2) + 4as_2m^2 ln(1+s_2m^2) \right] dm \]
Implies after reduce above integral with equal zero right side and assume \( a=1 \) we get:
\[ s_2 = \frac{145625}{246078} \]
So the approximate solution is \( \tilde{k}(m) = 1 + \frac{145625}{246078} m^2 \)

Similarly we can apply that steps if we take other intervals.

4- Discuss the Results and Conclusion

In [3] Chowdhury and Hashim find the approximate solution for the above examples but not drawn these solutions. In this section the compared between these approximate solutions and exact solution drawn in the same figure.

For example (1), the exact solution is \( u_0(x) = x^4 - x^3 \) and the approximate solution is \( \tilde{n}(m) = \frac{336689547}{47334880} m^2 \)

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So the following figure shown these solutions

For example (2), the exact solution is \( k(m) = e^{-\alpha m^2} \)
and the approximate solution is \( \tilde{k}(m) = 1 + \frac{145625}{246078} m^2 \)

So the following figure shown these solutions

**References:**

حل تقريبي لمعادلة إمدن فاولر باستخدام طريقة جاليركين

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الخلاصة:
معادلة إمدن–فاولر (E–F.Eq) تستخدم في الرياضيات مع العلوم الأخرى مثل الفيزياء والفيزياء الكيميائية والفيزياء الفلكية، كما يمكن احتزال هذه المعادلة إلى معادلة لين–إمدن بدالة محددة واستخدامها في العلوم المختلفة مع الرياضيات. درس العديد من المؤلفين الطرق التحليلية والعددية لإيجاد حل لهذا النوع من المعادلات في الحالة الخطية أو غير الخطية إحدى هذه الطرق وهي طريقة اضطراب التماثل. في هذا العمل تم إيجاد الحل التقريبي لمعادلات نوع إمدن–فاولر المعممة للمعادلات التفاضلية العادية من الدرجة الثانية بواسطة طريقة جاليركين وهي إحدى الطرق المتبقية الموزونة ولا تحتاج إلى وقت طويل وتساعد العامل (الخطي أو غير الخطي) أو العامل التفاضلي ومقارنة هذا الحل مع الحل النتائج من خلال تطبيق هذه الطرق على المعادلات المجاورة وغير المجاورة وإدراج الحلول في نفس النطاق لتوضيح النتائج.