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Piecewise Monotone Approximation of Unbounded Functions In Weighted Space $L_{P,w}([-\pi,\pi])$

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ABSTRACT

In this paper, investigate the approximation of unbounded functions in weighted space, by using trigonometric polynomials considered. We introduced type of polynomials piecewise monotone (q_κ) having same local monotonicity as unbounded functions without affecting the order of huge error have a finite number of max. and min. unbounded functions that amount. In addition, we established not included any of extreme points of this functions, χ of and closed subset γ on closed intervals χ then there exist class of polynomials such that the best of approximation has high or order of β and such that for κ sufficiently great of the polynomials and functions have the same monotonicity at each of γ.

1. INTRODUCTION

Let $\chi = [-\pi, \pi]$ be closed intervals, $L_P([-\pi, \pi])$we can be define as the space of all bounded function on χ, with the norms.

$$\|f\|_p = \left(\int_X |f(t)|^p dt\right)^{\frac{1}{p}} < \infty$$

Let $\mho(w)$ be the set of all weighted function on χ, a weighted function can be written. The follows $w : \chi \longrightarrow \mathcal{R}^+$ such that $\left| f(t) \right| \le \mathbb{M}/w(t)$, $w > 0, \mathbb{M} \in \mathcal{R}^+$, $w(t) \in \mho(w)$ and $L_{p,w}([-\pi,\pi])$, is the space of all unbounded functions on χ, which are equipped the norms.

$$\|f\|_{p,w} = \left(\int_X |f(t).w(t)|^p dt\right)^{\frac{1}{p}} < \infty$$

Let $\mathbb{T}_\kappa$ be the set Which contains any polynomials of degree less than or equal to κ.

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Let $[a, b]$ be a compact interval and $f : [a, b] \longrightarrow \mathcal{R}^+$ a function, f is piecewise monotone if and only if there exists a partition $a = t_1 < t_2 < t_3 < \cdots\cdots\cdots < t_k = b$, on $[a, b]$ and a, b will be called the topes of f, such that $f_i = f|_{(t_{i-1},t_i)}$ is monotone for all $i = 1,2 \ldots\ldots\ldots \kappa - 1$.

The aim of this paper is discuss piecewise monotone approximation of unbounded functions in weighted space by some types of efficient polynomials. An example[1], [2] and [3]. The strongest results are by [4], [5], [6] and [7]. In 2011, kopotun [8], introduce the point wise shape which preserving approximation of function by algebraic polynomials. In fact Dzyubenko [9] and Leviatan [10] have obtain interpolatory estimates in monotone piecewise polynomial approximation and both of them shown good result in monotone approximation.

So, we can define the operator as:

$$G_\kappa(t) = \frac{1}{g_\kappa}\left(\frac{\sin\kappa t/2}{\sin t/2}\right)^4; \text{ Where } \quad g_\kappa = \int_{-\pi}^{\pi}\left(\frac{\sin\kappa t/2}{\sin t/2}\right)^4 dt.$$

And $\kappa^3 \le g_\kappa \le 2\pi\kappa^3$.

2. Auxiliary Lemmas

Lemma 2.1. The operator $G_\kappa(t)$ has the following properties:

(i) If κ is even, $0 < \beta \le \pi$, then

$\int_{-\pi}^{\pi} G_\kappa(t)\,dt \le \int_0^{\beta} G_\kappa(t)dt$.

(ii) if, $0 < \beta \le \pi/2$, then

$\int_{\beta}^{\pi} G_\kappa(t)\,dt \le \frac{C}{\kappa^3\beta^3}$.

Where C is positive constant of real numbers.

Proof: (i) We take $0 < \beta \le \pi/2$;

$G_\kappa(t)$ is an even periodic and we have

$\int_{\pi}^{\pi+\beta} G_\kappa(t)\,dt = \int_{\pi-\beta}^{\pi} G_\kappa(t)\,dt$.

So , $\int_0^{\beta}[G_\kappa(t) - G_\kappa(t+\pi)]\,dt = \int_0^{\beta} G_\kappa(t)\,dt - \int_{\pi}^{\pi-+\beta} G_\kappa(t)\,dt$ …. (2.1)

Also, if $\pi/2 < \beta \le \pi$, Then

$\int_0^{\beta} G_\kappa(t)\,dt - \int_{\pi-\beta}^{\pi} G_\kappa(t)dt = \int_0^{\pi-\beta} G_\kappa(t)dt - \int_{\beta}^{\pi} G_\kappa(t)dt$ …. (2.2)

From (2.1) and (2.2) we obtain

$$\int_{-\pi}^{\pi} G_\kappa(t)\,dt \le \int_0^{\beta} G_\kappa(t)\,dt.$$

(ii) Let $0 < \beta \le \pi/2$. Then, since $\kappa^3 \le g_\kappa \le 2\pi\kappa^3$.

So, $\int_{\beta}^{\pi} G_\kappa(t)\,dt \le \frac{1}{\kappa^3}\int_{\beta}^{\pi}(\frac{\sin\kappa t/2}{\sin t/2})^4 dt \le \frac{1}{\kappa^3}\int_{\beta}^{\pi} \frac{1}{(\sin t/2)^4} dt$.

Now, for $0 < t \le \pi$. $\sin t/2 \ge t/2$, hence

$\int_{\beta}^{\pi} G_\kappa(t)\,dt \le \frac{\pi^4}{\kappa^3}\int_{\beta}^{\pi}\frac{1}{t^4}dt = \frac{\pi^4}{3\kappa^3}[\frac{-1}{\pi^4} + \frac{1}{\beta^3}] \le \frac{C}{\kappa^3\beta^3}$

And this lemma is proving.

3. Main results

Theorem 3.1 :

Let $t \in [-\pi,\pi], f \in L_{P,w}([-\pi,\pi]), 1 \le P < \infty$ Periodic function. Then there exist trigonometric polynomial q_κ of order leas than or equal to κ, such that $\|f - q_\kappa\|_{P,w} \le \frac{C}{\kappa}$, where C is positive constant.

Proof. we can choose $q_\kappa(t) = q_\kappa(t + 2\kappa\pi)$, $\kappa = 1,2 \dots\dots$ and

$q_\kappa(t) = \int_{-\pi}^{\pi} G_\kappa(b)f(t-b)\,db$

Since f Periodic function, we have

$q_\kappa(t) = \int_{-\pi}^{\pi} G_\kappa(b)f(t-b)\,db = \int_{t-\pi}^{t} G_\kappa(b)f(t-b)db - \int_{t}^{t+\pi} G_\kappa(b)f(t-b)\,db$

So,

$q'_\kappa(t) = \int_{t-\pi}^{t} G_\kappa(b)db - \pi f(t-\pi) - \int_{t}^{t+\pi} G_\kappa(b)\,db - \pi f(t+\pi)$

$= \int_{t-\pi}^{t} G_\kappa(b)db - \int_{t}^{t+\pi} G_\kappa(b)\,db$

We can take $0 < t \le \pi$, and $G_\kappa(b)$ is an periodicity. From Lemma 2.1, we obtain

$$q'_\kappa(t) = 2\int_0^{t} G_\kappa(b)db - 2\int_{\pi-t}^{\pi} G_\kappa(b)\,db$$

Since, $2\int_0^{t} G_\kappa(b)db \ge 2\int_{\pi-t}^{\pi} G_\kappa(b)\,db$

Implies $q'_\kappa(t) \ge 0$

Also, for $-\pi < t \le 0$, we obtained $q'_\kappa(t) \le 0$, $q_\kappa(t)$ non-negative and,

$|q_\kappa - f| \ge 0$, implies $\|q_\kappa - f\|_{p,w} \le \frac{C}{\kappa}$.

Theorem 3.2:

Let $q_\kappa(t)$ be a monotone function in $L_{P,w}\left(\left[-\frac{\pi}{2},\frac{\pi}{2}\right]\right)$. Then there exist a piecewise linear operator Ƒ, such that $\|q_\kappa - F\|_{p,w} \le \frac{C}{\kappa} Max(F)$, where C is positive constant.

Proof. Let $-\frac{\pi}{2} = t_1 < t_2 < t_3 < \dots\dots\dots < t_k = \frac{\pi}{2}$

We need to make the $q_\kappa \in \mathbb{T}_k$, such that satisfying

$$\|q_\kappa - F\|_{p,w} \le \frac{C}{\kappa} Max(F)$$

Let $k_0 = \frac{p_0+p_{\kappa-1}}{2}$, $k_i = \frac{p_i-p_{i-1}}{2}$; For $i = 1,2 \dots\dots, \kappa - 1$.

Thus,

$$F(t) = K + \sum_{i}^{\kappa-1} k_i \left|t - s_i\right| \quad \dots (3.1)$$

Where $-\frac{\pi}{2} = s_1 < s_2 < s_3 < \dots\dots\dots < s_k = \frac{\pi}{2}$ and Ҡ is constant.

Let q be defined by theorem (1) and, let

$$q_\kappa(t) = K + \sum_{i=0}^{\kappa-1} k_i\, q(t - s_i) \quad \dots (3.2)$$

Where $q_\kappa \in \mathbb{T}_k$. So, from (3.1), and (3.2), we get

$\|q_\kappa - F\|_{p,w} \le \frac{C}{\kappa}\, \mathrm{Max}\left|\sum_{i=0}^{\kappa-1} k_i\right| \le \frac{C}{\kappa}\,\mathrm{Max}(p_i) = \frac{C}{\kappa}\,\mathrm{Max}(F)$.

Theorem 3.3:

Let $f \in L_{P,w}\left(\left[-\frac{\pi}{2},\frac{\pi}{2}\right]\right), 1 \le P < \infty$ and C is positive constant independent of $\boldsymbol{k}$.Then there. exist a monotone function $< q_\kappa >, q_\kappa \in \mathbb{T}_k$ such that

$$\|f - q_\kappa\|_{P,w} \le \frac{Max(C)}{\kappa}.$$

Where f is the proper piecewise monotone functions.

Proof. Let F_k be the proper piecewise linear operator define on $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$,such that its has nodes at the topes of the function f at the points $-\frac{\pi}{2}+\frac{i\pi}{k}$, for $i = 0,1,2 \ldots\ldots, \kappa$, and $F_k(t) = f(t)$, since $\|F_\kappa - f\|_{P,w} \le \frac{C_1}{\kappa}$,

where $\mathcal{C}$ is positive constant and $Max(F) \le \frac{C_1}{\kappa}$. By using theorem (2), there is a Polynomial $q_\kappa \in \mathbb{T}_k$ such that

$$\|F_\kappa - q_\kappa\|_{p,w} \le \frac{\mathcal{C}_2}{\kappa}.$$

We need to prove that

$$\|f - q_\kappa\|_{p,w} \le \frac{Max(C)}{\kappa}.$$

So,

$$\|f - q_\kappa\|_{P,w} = \left(\int_X |[f(t) - q_\kappa(t)].w(t)|^p dt\right)^{\frac{1}{p}} =$$

$$\left(\int_X |[f(t) - F_k(t) + F_k(t) - q_\kappa(t)].w(t)|^p dt\right)^{\frac{1}{p}}$$

$$\le \left(\int_X |[f(t) - F_k(t)].w(t)|^p dt\right)^{\frac{1}{p}} + \left(\int_X |[F_k(t) - q_\kappa(t)].w(t)|^p dt\right)^{\frac{1}{p}}$$

$$\le Max\left[\left(\int_X |[f(t) - F_k(t)].w(t)|^p\, dt\right)^{\frac{1}{p}} + \left(\int_X |[F_k(t) - q_\kappa(t)].w(t)|^p dt\right)^{\frac{1}{p}}\right]$$

$$\le Max\left(\int_X |[f(t) - F_k(t)].w(t)|^p\, dt\right)^{\frac{1}{p}} + Max\left(\int_X |[F_k(t) - q_\kappa(t)].w(t)|^p dt\right)^{\frac{1}{p}}$$

$$\le Max(\mathcal{C})\, \|F_\kappa - f\|_{P,w} + Max(\mathcal{C})\|F_\kappa - q_\kappa\|_{p,w}$$

$$\le \frac{Max(\mathcal{C}_1)}{\kappa} + \frac{Max(\mathcal{C}_2)}{\kappa}$$

$$\le \frac{Max(C)}{k}.$$

Theorem 3.4 :

Let $f \in L_{P,w}([a,b]), 1 \le P < \infty$. Then there exist a nearly monotone operator $< p_\kappa >, p_\kappa \in \mathbb{P}_k$ such that

$$\|p_\kappa - f\|_{p,w} \le \frac{\mathcal{C}}{\kappa} Max(F).$$

Where f is the proper piecewise monotone on $[a,b]$.

Proof: The proof of this theorem by using the operator $q_\kappa(t) = \cos(t)$ with some modification. Of theorem 3.3

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أفضل التقريب الرتيب المتقطع للدوال الغير مقيدة في الفضاء الموزون

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الخلاصة:

في هذا البحث سنعرض تقريب الدوال الفضاء الوزون بواسطة متعددات مثلثية ونقدم نوع من المتعددات الرتيبة المتقطعة والتي لها نفس الرتابة الموقعية للدوال الغير مقيدة دون تاثير الخطا الكبير والتي يكون لها العدد المنتهي من الحدود العليا والدنيا اضافة الى ذلك عدم تضمين النقاط المتطرفة لهذه الدوال في الفترات الجزئية المغلقة للفترة الاساسية المغلقة لذا سوف سيتولد لنا فئة من متعددات الحدود من خلالها نحصل على أفضل تقريب من رتبة بيتا بيحث يكون لاي (ك) يكون لهذه الدوال والمتعددات نفس الرتابة.